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Money and Limited Enforcement in Multilateral Exchange

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Abstract

We propose a model in which money performs an essential role in the process of exchange, despite the presence of a multilateral clearing house. Agents are assumed to be anonymous and unable to make binding commitments. The clearing house can detect deviations but it cannot identify the individual deviator, hence, it punishes all traders collectively. The records of past deviations can be kept for a limited amount of time, after which they are wiped out. These features are enough to make room for a record-keeping device, such as money, that strictly improves the functioning of the clearing house.

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1 Introduction

The integration of monetary and value theory in a Walrasian general equilibrium framework, initially advocated by John Hicks, has turned out to be an enduring challenge. Frank Hahn (1973) has formulated the challenge in its starkest form, requiring money to help obtain market outcomes that could not otherwise be achieved, *i.e.* requiring money to be essential. Unfortunately, the Walrasian general equilibrium model has a clearing facility and contractual arrangements that work too smoothly for money to have any socially beneficial role in lubricating the process of trade. To carve up an essential role for money, monetary theorists have built, over the years, several ingenious models with spatial separation, bilateral trade and informational frictions[1] to impede as much as possible the functioning of the Walrasian clearing facility and its contractual arrangements. Absent a clearing house, the inability to make binding commitments and the anonymity of agents, have been shown to be necessary ingredients for money to be essential (Narayana Kocherlakota (1998)). We propose a model where the agents are anonymous and cannot commit themselves to future actions, but trade can be facilitated by a centralized clearing facility with limited enforcement power. We use the model to find out whether money can improve the functioning of the clearing house.

Specifically, we consider a model in which consumption smoothing in the face of a jagged, deterministic income stream constitutes the motive for a finite number of agents to trade their endowments as in Townsend (1980). Differently from Townsend (1980), however, markets are not spatially separated and the agents can access a clearing facility at any point in time. In keeping with the spirit of the Walrasian approach, agents make contact only and directly with the market facility. The clearing facility pools and redistributes the endowments, without knowing the identities of the agents. The anonymity of the agents, while limiting the operation of the clearing

[1] *e.g.* the turnpike model by Townsend (1980) or the random matching model of money by Nobuhiro Kiyotaki and Randall Wright (1989, 1991, 1993).

house, does not impede it altogether. Shortfalls in the contribution to the pool can be detected, but the blame cannot be pinned on any specific agent and no individualized punishment can be administered. When a shortfall is detected, the clearing facility shuts down, thus leaving everybody in autarky. The record of any deviation can be kept only for a limited amount of time, after which it is completely wiped-out and the facility is opened up again. This captures the inability of the clearing house to perfectly enforce punishments.[2] This also restricts the set of allocations that can be achieved, but, again, it does not impede trade completely. As regards withdrawals from the pool, the facility cannot tailor the amounts it makes available to the needs of any specific type of agent, being unable to identify them.

We consider two regimes for the clearing house, without and with fiat money. The clearing facility is quite effective even without money. In fact, in some cases, even allocations that are on the Pareto frontier can be reached. With fiat money, contributions to the pool are rewarded with cash, withdrawals from the pool must be paid for in cash. The facility can operate, in this case, a transfer scheme, to control the deflation or inflation rate. Because of anonymity, transfers must be identical for all agents. The agents may refuse to participate in the transfer scheme, when it entails negative transfers - i.e. taxes. Should this happen, the facility would close down. It can, however, in this case as well, be opened up again after a wipe-out of the records, in a way which is symmetric with respect to the case without money. We provide a complete characterization of the allocations and their ex-ante welfare properties under the two regimes.

Our main result consists in the comparison between the two regimes. First, we prove that the cash-based facility can always attain at least the same set of allocations as the cash-less one, and, from the point of view of ex-ante welfare, the former does always at least as well as the latter. The cash-based system is more flexible than the cash-less one, since it allows agents to optimize against their budget constraints,

[2] We thank Randall Wright for suggesting this way of capturing limited enforcement.

subject to a participation constraint which applies only for the decision to pay taxes. The cash-less system, instead, needs to ensure participation for all transactions. Next, we find the conditions under which money dominates strictly. The cash-based facility performs strictly better than the cash-less one, only if the records are wiped out in finite time, i.e. the ability of the clearing house to enforce punishments is limited. When the records are wiped out fast enough, and, thus, enforcement is sufficiently imperfect, money strictly improves the functioning of the clearing house.

The literature on the essentiality of money has applied mechanism design to economies characterized by some form of spatial separation, implying that trade cannot be conducted via a multilateral clearing house, and informational imperfections. Prominent examples are Kocherlakota (1998) and Wallace (2011). Our work features similar informational imperfections - namely, anonymity and limited commitment, but not the spatial ones. The assumption that the records of deviations are wiped out after some time is akin to the one adopted in Kocherlakota and Wallace (1998), where a record of past actions is updated with an exogenous time lag. David Levine (1991) features an environment with anonymity in a Walrasian setting. The notion of anonymity adopted is stronger than ours. In Levine (1991) individual deviations cannot be detected. This precludes the working of a non-monetary system from the start. In our paper, with a finite number of agents, individual deviations can be detected, and the non-monetary system can operate, although only imperfectly. Our results speak to the strand of literature originated from the work of Ricardo Lagos and Randall Wright (2005), by now known as the *New Monetarist* approach to monetary theory, which features alternating bilateral and multilateral trading sessions. Our model can be seen as a version of their environment with multilateral trading sessions occurring all the time, akin to one of the models appearing in Guillaume Rocheteau and Wright (2005). Our result suggests that, in a finite economy with multilateral clearing, the essentiality of money requires the clearing house to be unable to perfectly enforce the threat of punishment of deviations. Charalambos

Aliprantis, Gabriele Camera and Daniela Puzzello (2007a,b) have argued that the *New Monetarist* approach needs to carefully specify the process that determines the bilateral meetings, lest money turn out to be inessential. We suggest that with multilateral clearing, the crucial aspect for the essentiality of money is, instead, limited enforcement. Joseph Ostroy and Ross Starr (1974) ask how a Walrasian equilibrium allocation may be reached through a sequence of bilateral trades subject to limited commitment and informational constraints. We share the relevance of the inability to commit and of the informational frictions as constraints for the execution of trades, but we adopt a multilateral framework with a clearing house. In a series of papers, Nobu Kiyotaki and John Moore (e.g. Kiyotaki and Moore (2002)) have developed a theory of liquidity and money based on the limited ability of agents to make bilateral and multilateral commitments. Our model also places commitment centre stage, but in a context where a centralized clearing system is at work.

The rest of the paper is organized as follows. Section 2 presents the model. Section 3 characterizes the first best allocations. Section 4 analyzes the non-monetary system. Section 5 the monetary one. Section 6 compares the two and shows when money is essential. Section 7 concludes. In the Appendix, we discuss a transfer scheme for the monetary regime that can Pareto improve upon the one presented in the main body of the paper.

2 Fundamentals

Time, indexed by $t = 1, 2, ...$, is discrete and continues for ever. There is a single perishable good, x. The economy is populated by $2N$ agents, equally divided between two types, indexed by $i = 1, 2$. Agents of type 1 receive, as an endowment, one unit of the good at odd dates and zero at even dates, i.e. $e_t^1 \in \{0, 1\}$, $e_t^1 = 0$ for $t = 2n$, $e_t^1 = 1$ for $t = 2n - 1$, with $n \in \mathbb{N}$, and agents of type 2 receive $e_t^2 = 1 - e_t^1$ for all $t \geq 1$. Agents' preferences over consumption of the good are represented by the

following life-time utility

$$\sum_{t=1}^{\infty}\beta^{t-1}u\left(x_t^i\right),$$

where $\beta \in (0,1)$ represents the discount rate and $u\left(x_t^i\right)$ the period utility function defined over $x_t^i \in \mathbb{R}_+$, the units of the good consumed by an agent of type i at date t. The function $u : \mathbb{R}_+ \to \mathbb{R}$, is (at least twice) continuously differentiable, with $u'(x) > 0$, $u''(x) < 0$, for all $x \in \mathbb{R}_+$, and $u'(0) = +\infty$. We also require $u(0) > -\infty$.

Agents cannot commit themselves to future actions and are anonymous, i.e. their identities are private information.

3 First Best Allocations

We begin with a characterization of the first best allocations. Let $\mu \in [0,1]$ be the weight given to the well-being of type 1 agents. The first best allocations are those that maximize the weighted sum of life-time utilities of the two types

$$\underset{\{x_t^1\}_{t=1}^{\infty},\{x_t^2\}_{t=1}^{\infty}}{Max} \quad \mu\left(\sum_{t=1}^{\infty}\beta^{t-1}u\left(x_t^1\right)\right)+(1-\mu)\left(\sum_{t=1}^{\infty}\beta^{t-1}u\left(x_t^2\right)\right), \tag{1}$$

subject to feasibility,

$$Nx_t^1 + Nx_t^2 \leq N, \forall t \geq 1, \tag{2}$$

and non-negativity $x_t^i \geq 0$, $\forall t \geq 1, \forall i$. The feasibility constraint, (2), can be taken at equality, since the objective function is strictly increasing in the choice variables for all $\mu \in [0,1]$. Clearly, it is optimal to assign $x_t^1 = 0$ and $x_t^2 = 1$, $\forall t \geq 1$, when $\mu = 0$, and viceversa for $\mu = 1$. Consider now $\mu \in (0,1)$. We can ignore the non-negativity constraints on consumption for any $\mu \in (0,1)$, thanks to the Inada condition. Using (2) at equality into the objective function, (1), we can compute the necessary - and sufficient, given the strict concavity of the utility function- condition for an optimum as

$$\Omega\left(x_t^1,\mu\right) \equiv \mu u'\left(x_t^1\right) - (1-\mu)\,u'\left(1-x_t^1\right) = 0, \forall t \geq 1, \tag{3}$$

together with $x_t^2 = 1 - x_t^1, \forall t \geq 1$. Clearly from (3), the optimal x_t^1 (and therefore x_t^2) is constant over time, $x_t^1 = x^1, \forall t \geq 1$. Let $z : [0,1] \to [0,1]$ be the function $z(\mu)$ that identifies the first best allocation for type 1 for all values of $\mu \in [0,1]$, and $1 - z(\mu)$ the first best allocation for type 2. The following Lemma gives a complete characterization of $z(\mu)$. Notice that the symmetric allocation is on the Pareto frontier. This observation will turn out to be useful later on.

Lemma 1 *a. For any $\mu \in (0,1)$, there exists a unique $z \in (0,1)$ such that $\Omega(z,\mu) = 0$; b. The function $z(\mu)$ is (at least once) continuously differentiable in μ, with $\frac{\partial z(\mu)}{\partial \mu} > 0$, for any $\mu \in (0,1)$; c. $z(0) = 0$, $z\left(\frac{1}{2}\right) = \frac{1}{2}$ and $z(1) = 1$.*

Proof. a. $\Omega(x^1, \mu)$ is (at least once) continuously differentiable in x^1, with $\Omega(0,\mu) = +\infty$, and $\Omega(1,\mu) = -\infty$. By the Intermediate Value Theorem a value $z \in (0,1)$ such that $\Omega(z,\mu) = 0$ exists and is unique, since $\frac{\partial \Omega\left(x^1,\mu\right)}{\partial x^1} = \mu u''(x^1) + (1-\mu) u''(1-x^1) < 0$ for all $\mu \in (0,1)$ and $x^1 \in [0,1]$. b. By the Implicit Function Theorem, $z(\mu)$ is (at least once) continuously differentiable in μ, with $\frac{\partial z(\mu)}{\partial \mu} = -\frac{\frac{\partial \Omega\left(x^1,\mu\right)}{\partial \mu}}{\frac{\partial \Omega\left(x^1,\mu\right)}{\partial x^1}}\|_{x^1=z} > 0$, since $\frac{\partial \Omega\left(x^1,\mu\right)}{\partial \mu} = u'(x^1) + u'(1-x^1) > 0$ for all $\mu \in (0,1)$ and $x^1 \in [0,1]$; c. $z(0) = 0$ and $z(1) = 1$, since a type with zero weight in the objective function should be assigned zero consumption at an optimum; $\Omega\left(x_t^1, \frac{1}{2}\right) = \frac{1}{2}\left[u'(x_t^1) - u'(1-x_t^1)\right] = 0 \Leftrightarrow x_t^1 = 1 - x_t^1$, hence, $z\left(\frac{1}{2}\right) = \frac{1}{2}$. ■

Consider, for a moment, an economy in which agents could commit themselves to any future action. Given the utility function described in section 2, the First and Second Fundamental Welfare Theorems would apply. As a consequence, all the first best allocations identified by the function $z(\mu)$ could be decentralized as a competitive equilibrium. Agents would deliver their negative excess demands and withdraw their positive excesses through a clearing house that would be able to execute trade without impediments. Since the ability of the agents to commit to future actions in our framework is limited, the actual realization of the equilibrium allocations is not trivial. In what follows, we will describe the workings of a multilateral clearing house, first

without and, then, with the use of fiat money to facilitate the execution of trades.

4 Non-Monetary Regime

The Clearing House We consider, first, a transaction technology that does not make use of cash. We will refer to it as the non-monetary, or cash-less, facility. The non-monetary facility operates, at each date t, following a two-stage procedure. 1. The facility requires agents to deliver an amount $d \in [0,1]$ of the good; 2. the facility allows any agent to withdraw an amount $w \in \{w_1, w_2\}$ of the good, whereby an agent of type i obtains w_i with probability $\psi \in [0,1]$ and $w_{j\neq i}$ with the complementary probability. The facility operates subject to feasibility, i.e.

$$w_1 N + w_2 N \leq dN. \tag{4}$$

If, at any point in time, any delivery at stage 1 is smaller than d, the facility stops without moving to stage 2. The clearing house can keep the records of any deviation only for a limited amount of time. After T periods, with $T = 2n$, $n \in \mathbb{N} \cup \{0\}$, the records are completely wiped out.[3] Following such a loss of information, the facility can be re-started. Once re-started, the facility operates as before. Define the vector $\omega = (w_1, w_2)$. The facility, before the beginning of trade, at date 0, is programmed with the parameters (d, ω, ψ) to maximize ex-ante welfare and ensure participation by the agents, taking into account that they are anonymous. Its workings are common knowledge at date 0. Notice that, given the symmetry and repetitiveness of the environment, where agents belonging to the same type are identical and receive identical endowments every other period, we restrict attention to facilities that treat agents identically by type and over time, i.e. we look at allocations that are symmetric and

[3] The assumption that the records are wiped out after an even number of periods seems appropriate in this environment where agents go through period-two cycles. It leads to a tighter participation constraint than the alternative possibility with an odd number of periods. The participation constraint for an odd number of periods coincides always with the one with $T = \infty$.

stationary. This is consistent with our focus on allocations that maximize ex-ante welfare.

Deliveries and withdrawals The first result comes immediately from the assumption that agents are anonymous, and, thus, at stage 2, the facility cannot distinguish between an agent of type 1 or 2. We immediately have

Lemma 2 *Since agents are anonymous,* $\psi = \frac{1}{2}$.

Proof. The amount w_1 is assigned to agents of type 1 with probability ψ and to agents of type 2 with probability $1-\psi$ (and viceversa for w_2). Since agents' identities are unknown to the facility, $\psi = 1 - \psi$. ■

Another consequence of the anonymity of the agents has been already built into the way the facility works following a deviation at stage 1. The facility can detect whether somebody delivered less than d, but it cannot tell the identity of the deviator. Hence, it cannot punish directly a single individual. It can, however, punish all the agents simultaneously shutting down trade altogether.

The length of time, T, after which the records are wiped out and the facility opens up again, parameterizes the severity of the consequences of the lack of enforcement power by the clearing house. If $T = 0$, trade will resume so fast that deviators cannot be punished, and the consequences of the lack of enforcement are most severe. At the other extreme, for $T \to \infty$, the facility can never be re-opened, thus making the threat of punishment most effective.

Define $f_T(\beta) \equiv \sum_{j=1}^{\frac{T}{2}} \beta^{2j-1} = \beta\left(\frac{1-\beta^T}{1-\beta^2}\right)$ and $g_T(\beta) \equiv \sum_{j=0}^{\frac{T}{2}} \beta^{2j} = \frac{1-\beta^{T+2}}{1-\beta^2}$. For a general T, to ensure participation by agents who are in a position of delivering some amount of the good at any given date, the parameters (d, ω) will have to satisfy the

following constraint

$$g_T(\beta)\left[\frac{1}{2}u(1-d+w_1)+\frac{1}{2}u(1-d+w_2)\right]+ \tag{5}$$
$$f_T(\beta)\left[\frac{1}{2}u(w_1)+\frac{1}{2}u(w_2)\right]\geq g_T(\beta)u(1)+f_T(\beta)u(0),$$

where the RHS is the payoff arising from the shut down of trade of T periods following a deviation at some date, and the LHS is the expected utility of abiding by the rules of trade for the same length of time.[4] The participation condition (5) is written from the point of view of an agent who is supposed to deliver some amount d at the current date, assuming that all other agents are always participating in the trading arrangement. Should the agent expect all other agents not to participate at any point in time, his best response would be not to deliver anything. Hence, autarky is always an equilibrium of this trading arrangement. Notice that, due to the symmetry and stationarity of the environment, every agent, whenever in a position to decide whether to deliver some amount of the good, is confronted with the same participation condition (5). Notice also that, after the records are wiped out, all agents would agree to the re-opening of the clearing house, provided (5) holds.

Lemma 3 *Following a deviation, after T periods, if (5) holds, then, all agents would agree to a re-opening of the clearing house.*

Proof. T periods after a deviation, when the records are wiped out, either an agent is in a position to deliver something or not. In the former case, an agent can always accept the re-opening and, then, refuse to deliver. If (5) holds, participation of this type of agent is ensured. In the latter case, the agent would always agree to a re-opening. ■

[4] After the facility re-starts the payoff reverts to the same value, hence, it drops out of the inequality.

Ex-ante welfare, for $\mu \in [0,1]$, which represents the weight of type 1 agents in the welfare function, is given by

$$W(d,\omega,\mu,\beta) = \frac{1}{2\left(1-\beta^2\right)} \left[\begin{array}{c} \mu\left(\sum_{i=1}^{2} u(1-d+w_i) + \beta \sum_{i=1}^{2} u(w_i)\right) + \\ (1-\mu)\left(\sum_{i=1}^{2} u(w_i) + \beta \sum_{i=1}^{2} u(1-d+w_i)\right). \end{array} \right] \quad (6)$$

Strict concavity of the utility function implies that, given a random variable w with non-degenerate distribution, $u(E(c+w)) > Eu(c+w)$ for any $c \in \mathbb{R}_+$, where E is the expectation operator. Therefore, we have the following result,

Lemma 4 *For any $d \in [0,1]$ and $\mu \in [0,1]$, $w_1 = w_2 = \frac{d}{2}$ in order to maximize (6) subject to (5) and (4).*

Proof. By strict concavity of the utility function, both the LHS of (5) and (6) are strictly higher with $w_1 = w_2 = w$, for any $d \in [0,1]$ and $\mu \in [0,1]$. Since the utility function is strictly increasing, wasting resources can only decrease the agents' welfare, hence, we have, from (4) taken at equality, $w_1 = w_2 = \frac{d}{2}$. ■

Thus, in order to work in the best interest of the agents, while satisfying participation and feasibility, the facility should set

$$w_1 = w_2 = \frac{d}{2}. \quad (7)$$

With (7), the participation constraint (5) becomes

$$g_T(\beta)\, u\left(1-\frac{d}{2}\right) + f_T(\beta)\, u\left(\frac{d}{2}\right) \geq g_T(\beta)\, u(1) + f_T(\beta)\, u(0), \quad (8)$$

and (6) reduces to

$$\mu\left[u\left(1-\frac{d}{2}\right) + \beta u\left(\frac{d}{2}\right)\right] + (1-\mu)\left[u\left(\frac{d}{2}\right) + \beta u\left(1-\frac{d}{2}\right)\right]. \quad (9)$$

Hence, we have reduced the problem of the clearing house to the choice of $d \in [0,1]$, so as to maximize (9) and satisfy (8).

We begin with a characterization of the allocations that satisfy the participation constraint (8), which we will call sustainable. Then, we will choose, among the sustainable allocations, those that maximize the agents' ex-ante welfare (9).

Sustainable Allocations We call sustainable, the allocations satisfying the participation constraint.

Definition 1 *An allocation that satisfies the participation constraint (8) is sustainable without cash.*

For the purpose of comparing the current system with the monetary one, it is convenient to make a change of variable, defining $y \equiv 1 - \frac{d}{2} \in \left[\frac{1}{2}, 1\right]$. Although only values of $y \geq \frac{1}{2}$ are feasible, it will be more convenient to consider $y \in [0,1]$, as a first step. Let us define $y_T(\beta) : (0,1) \to [0,1]$ as the function that explicitly relates pairs of values $(y, \beta) \in [0,1] \times (0,1)$ such that $\Psi_T(y, \beta) = 0$, for any given T. We will then restrict y to the feasible interval $\left[\frac{1}{2}, 1\right]$. Define the function $\widehat{y}_T(\beta) : (0,1) \to \left[\frac{1}{2}, 1\right]$, as $\widehat{y}_T(\beta) \equiv \max\left\{y_T(\beta), \frac{1}{2}\right\}$, for any $\beta \in (0,1)$ and $T > 0$, i.e. the restriction of $y_T(\beta)$ to feasible values in $\left[\frac{1}{2}, 1\right]$. Let the function $\Psi_T(y, \beta) : [0,1] \times (0,1) \to \mathbb{R}$ be defined as

$$\Psi_T(y, \beta) \equiv f_T(\beta)\left[u(1-y) - u(0)\right] - g_T(\beta)\left[u(1) - u(y)\right], \tag{10}$$

as the difference between the LHS and the RHS of (8). Finally, let $\Gamma_T(\beta) : (0,1) \rightrightarrows \left[\frac{1}{2}, 1\right]$ be the correspondence that satisfies $\Psi_T(y, \beta) \geq 0$ for any given $T > 0$, identifying sustainable allocations as β varies over its domain of definition.

Lemma 5 *An allocation y is sustainable without cash if and only if $y \in \Gamma_T(\beta)$*

Proof. i. "if" part. By definition $y \in \Gamma_T(\beta)$ satisfies $\Psi_T(y, \beta) \geq 0$, i.e. (8). Hence, it is sustainable without cash according to Definition 1. ii. "only if" part. Suppose $y \notin \Gamma_T(\beta)$. Then, it violates $\Psi_T(y, \beta) \geq 0$, i.e. (8). Hence, it is not sustainable without cash according to Definition 1. ■

We begin our characterization of sustainable allocations with the no-trade, autarkic ones, i.e. the allocation $y = 1$. No-trade allocations are obviously always sustainable.

Lemma 6 $1 \in \Gamma_T(\beta)$ *for any* $\beta \in (0,1)$ *and* $T \geq 0$.

Proof. Substituting $y = 1$ into (10), we have $\Psi_T(1, \beta) = 0$ for any $\beta \in (0, 1)$ and $T \geq 0$. ■

When the facility can be restarted immediately, i.e. $T = 0$, the only allocation that can be sustained is the autarkic one. This occurs because the punishment for failing to deliver the good is ineffective.

Lemma 7 *$\Gamma_0(\beta) = \{1\}$ for any $\beta \in (0, 1)$.*

Proof. With $T = 0$, $\Psi_0(y, \beta) = -[u(1) - u(y)]$. $\Psi_0(y, \beta) \geq 0 \Leftrightarrow y = 1$ for any $\beta \in (0, 1)$, since the utility function is monotonic and $y \leq 1$. ■

Next, we investigate whether there are non-autarkic allocations ($y < 1$) that can be sustained for $T > 0$. The following Lemma provides an intermediary step towards the characterization of the sustainable non-autarkic allocations.

Lemma 8 *a. For any $\beta \in (0, 1)$ and $T > 0$, there exists exactly one value $y \in [0, 1)$ s.t. $\Psi_T(y, \beta) = 0$. b. For any $T > 0$, the function $y_T(\beta)$ is (at least twice) continuously differentiable in β, with $\frac{\partial y_T(\beta)}{\partial \beta} < 0$ for all $\beta \in (0, 1)$.*

Proof. a. The function $\Psi_T(y, \beta) : [0, 1] \times (0, 1) \to \mathbb{R}$ is (at least twice) continuously differentiable in y. Observe that, for any $T > 0$, $\Psi_T(0, \beta) = -\frac{1+\beta^{T+1}}{1+\beta}[u(1) - u(0)] < 0$, $\Psi_T(1, \beta) = 0$, $\frac{\partial \Psi_T(y,\beta)}{\partial y} = -\frac{\beta\left(1-\beta^T\right)}{1-\beta^2}u'(1-y) + \frac{1-\beta^{T+2}}{1-\beta^2}u'(y)$, $\frac{\partial \Psi_T(0,\beta)}{\partial y} = +\infty$, $\frac{\partial \Psi_T(1,\beta)}{\partial y} = -\infty$, $\frac{\partial^2 \Psi_T(y,\beta)}{\partial y^2} = \frac{\beta\left(1-\beta^T\right)}{1-\beta^2}u''(1-y) + \frac{1-\beta^{T+2}}{1-\beta^2}u''(y) < 0$. Hence, for any $\beta \in (0, 1)$ and $T > 0$, there is exactly one $y \in [0, 1)$ s.t. $\Psi_T(y, \beta) = 0$. b. The derivative $\frac{\partial \Psi_T(y,\beta)}{\partial y}$ evaluated at any $(y, \beta) \in (0, 1) \times (0, 1)$ such that $\Psi_T(y, \beta) = 0$ is given by

$$g_T(\beta)[u(1) - u(y)]\left[\frac{u'(y)}{u(1) - u(y)} - \frac{u'(1-y)}{[u(1-y) - u(0)]}\right] > 0, \tag{11}$$

since $\frac{u'(y)}{u(1)-u(y)} > \frac{1}{1-y} > \frac{u'(1-y)}{[u(1-y)-u(0)]}$ for any $y \in (0, 1)$ by strict concavity of the utility function. Therefore, the Implicit Function Theorem applies and $y_T(\beta)$ is (at

least twice) continuously differentiable in β. The derivative $\frac{\partial \Psi_T(y,\beta)}{\partial \beta}$ evaluated at any $(y,\beta) \in (0,1) \times (0,1)$ such that $\Psi_T(y,\beta) = 0$ is given by

$$g_T(\beta)\left[u(1) - u(y)\right]\left[\frac{f_T'(\beta)}{f_T(\beta)} - \frac{g_T'(\beta)}{g_T(\beta)}\right] > 0, \tag{12}$$

since $f_T'(\beta) = \sum_{j=1}^{\frac{T}{2}} (2j-1)\beta^{2j-2} > \sum_{j=0}^{\frac{T}{2}} 2j\beta^{2j-1} = g_T'(\beta)$ and $g_T(\beta) = \frac{1-\beta^{T+2}}{1-\beta^2} > \frac{\beta\left(1-\beta^T\right)}{1-\beta^2} = f_T(\beta)$. Thus, $\frac{\partial y_T(\beta)}{\partial \beta} < 0$, for any $\beta \in (0,1)$ and $T > 0$, since it is given by the ratio (12) to (11) changed of sign, as an application of the Implicit Function Theorem. ■

The next two Lemmas identify all the sustainable allocations for any $T > 0$. We begin with the case $T = \infty$. Some non-autarkic allocations - and sometimes even the symmetric allocation, which lies on the Pareto frontier- can be sustained. Let $\underline{\beta} \equiv \frac{u(1)-u\left(\frac{1}{2}\right)}{u\left(\frac{1}{2}\right)-u(0)} \in (0,1)$.

Lemma 9 *$\Gamma_\infty(\beta) = [\widehat{y}_\infty(\beta), 1]$ for all $\beta \in (0,1)$, where $\widehat{y}_\infty(\beta)$ is continuous, and*

i. $\widehat{y}_\infty(\beta) = \frac{1}{2}$, if $\beta \geq \underline{\beta}$;

ii. $\widehat{y}_\infty(\beta) = y_\infty(\beta) \in \left(\frac{1}{2}, 1\right)$, if $\beta < \underline{\beta}$.

Proof. The value $\underline{\beta}$ satisfies $\Psi_\infty\left(\frac{1}{2}, \beta\right) = \frac{1}{1-\beta^2}\left[\beta\left[u\left(\frac{1}{2}\right) - u(0)\right] - \left[u(1) - u\left(\frac{1}{2}\right)\right]\right] = 0$. Thus, $y_\infty\left(\underline{\beta}\right) = \frac{1}{2}$. By Lemma 8, $y_\infty(\beta)$ is (at least twice) continuously differentiable and strictly decreasing function for any $\beta \in (0,1)$. Hence, we have that for $\beta \in \left(0, \underline{\beta}\right)$, $y_\infty(\beta) > \frac{1}{2}$, $\lim_{\beta \to \underline{\beta}^-} y_\infty(\beta) = \frac{1}{2}$ and for $\beta \in \left[\underline{\beta}, 1\right)$, $y_\infty(\beta) \leq \frac{1}{2}$. By definition $\widehat{y}_T(\beta) \equiv \max\left\{y_T(\beta), \frac{1}{2}\right\}$. Thus, we have

$$\widehat{y}_\infty(\beta) = \begin{cases} \frac{1}{2}, & \text{if } \beta \geq \underline{\beta} \\ y_\infty(\beta), & \text{if } \beta < \underline{\beta} \end{cases},$$

which is continuous in $\beta \in (0,1)$, since $\lim_{\beta \to \underline{\beta}^-} y_\infty(\beta) = \frac{1}{2}$. ■

In this case, we have the harshest possible punishment, and all the feasible allocations can be sustained when agents are sufficiently patient, while only a strict

subset of the feasible allocations can be sustained when agents are impatient enough. Indeed, if the agents are more patient, the threat of a future punishment is harsher, thus extending the set of sustainable allocations. Figure 1 depicts $\Gamma_\infty(\beta)$ for any $\beta \in (0,1)$.

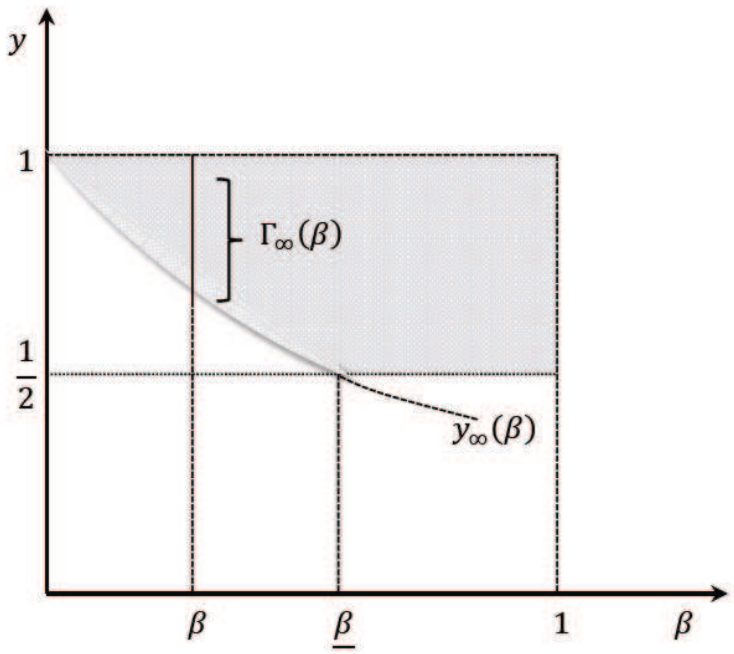

Figure 1: Sustainable Allocations, $T = \infty$

Next, we consider all the intermediate cases, with finite and positive T. Also in this case, some non-autarkic allocations - and even the Pareto efficient symmetric allocation, in some cases- can be sustained by the non-monetary facility. Define $\overline{T} \equiv \frac{2\left[u(1)-u\left(\frac{1}{2}\right)\right]}{2u\left(\frac{1}{2}\right)-u(1)-u(0)} \in (0,\infty)$.

Lemma 10 *For $0 < T < \infty, \Gamma_T(\beta) = [\widehat{y}_T(\beta), 1]$ for all $\beta \in (0,1)$, where $\widehat{y}_T(\beta)$ is continuous, and*

a. if $T > \overline{T}$, there exists a unique $\underline{\beta}_T \in (0,1)$ such that:

i. $\widehat{y}_T(\beta) = \frac{1}{2}$, for $\beta \geq \underline{\beta}_T$;

ii. $\widehat{y}_T(\beta) = y_T(\beta) \in \left(\frac{1}{2}, 1\right)$, for $\beta < \underline{\beta}_T$;

b. if $T \leq \overline{T}$, for all $\beta \in (0,1)$, $\widehat{y}_T(\beta) = y_T(\beta) \in \left(\frac{1}{2}, 1\right)$.

Proof. a. Evaluate $\Psi_T(y,\beta)$ at $y = \frac{1}{2}$ and $\beta \to 1$, obtaining $\lim_{\beta\to 1}\Psi_T\left(\frac{1}{2},\beta\right) = \left(\frac{T}{2}\right)\left[u\left(\frac{1}{2}\right) - u(0)\right] - \left(\frac{T}{2}+1\right)\left[u(1) - u\left(\frac{1}{2}\right)\right]$. Since $T > \overline{T}$, $\lim_{\beta\to 1}\Psi_T\left(\frac{1}{2},\beta\right) > 0$. Evaluate $\Psi_T(y,\beta)$ at $y = \frac{1}{2}$ and $\beta \to 0$, obtaining $\lim_{\beta\to 0}\Psi_T\left(\frac{1}{2},\beta\right) = -\left[u(1) - u\left(\frac{1}{2}\right)\right] < 0$. Since $\Psi_T\left(\frac{1}{2},\beta\right)$ is continuous in β, by the Intermediate Value Theorem there exists a value $\underline{\beta}_T \in (0,1)$ that solves $\Psi_T\left(\frac{1}{2},\beta\right) = 0$. The derivative $\frac{\partial \Psi_T\left(\frac{1}{2},\beta\right)}{\partial \beta} = f_T'(\beta)\left[u\left(\frac{1}{2}\right) - u(0)\right] - g_T'(\beta)\left[u(1) - u\left(\frac{1}{2}\right)\right] > 0$, since $f_T'(\beta) > g_T'(\beta)$ and $u\left(\frac{1}{2}\right) - u(0) > u(1) - u\left(\frac{1}{2}\right)$ by strict concavity of the utility function. Hence, $\underline{\beta}_T$ is unique. i. By Lemma 8, $y_T(\beta)$ is continuously differentiable with $\lim_{\beta\to\underline{\beta}_T^-} y_T(\beta) = \frac{1}{2}$, $\lim_{\beta\to 0^+} y_T(\beta) = 1$ and $\frac{\partial y_T(\beta)}{\partial \beta} < 0$. Hence, for $\beta \in \left[\underline{\beta}_T, 1\right)$, $y_T(\beta) \leq \frac{1}{2}$ and $\widehat{y}_T(\beta) = \frac{1}{2}$. ii. If $\beta \in \left(0, \underline{\beta}_T\right)$, once again by Lemma 8, $y_T(\beta) \in \left(\frac{1}{2}, 1\right)$ and $\widehat{y}_T(\beta) = y_T(\beta)$. Therefore, by definition of $\widehat{y}_T(\beta)$,

$$\widehat{y}_T(\beta) = \begin{cases} \frac{1}{2}, & \text{if } \beta \geq \underline{\beta}_T \\ y_T(\beta), & \text{if } \beta < \underline{\beta}_T \end{cases}$$

which is continuous in $\beta \in (0,1)$, since $\lim_{\beta\to\underline{\beta}_T^-} y_T(\beta) = \frac{1}{2}$. b. Since $T \leq \overline{T}$, $y = \frac{1}{2}$ never satisfies the participation constraint for any $\beta \in (0,1)$. A solution of $\Psi_T(y,\beta) = 0$ in $y \in \left(\frac{1}{2}, 1\right)$ exists for any $\beta \in (0,1)$, by the Intermediate Value Theorem, and is unique by the same argument used in part a. By Lemma 8, $y_T(\beta)$ is continuously differentiable in β with $\lim_{\beta\to 1^-} y_T(\beta) \geq \frac{1}{2}$, $\lim_{\beta\to 0^+} y_T(\beta) = 1$ and $\frac{\partial y_T(\beta)}{\partial \beta} < 0$. Hence, for all $\beta \in (0,1)$, $y_T(\beta) \in \left(\frac{1}{2}, 1\right)$. Therefore, by definition of $\widehat{y}_T(\beta)$, $\widehat{y}_T(\beta) = y_T(\beta)$, for all $\beta \in (0,1)$. ■

For all the intermediate cases, there are two possibilities. If the records are wiped out sufficiently infrequently and, thus, the threat of future punishment is severe enough, the situation is fairly similar to the previous case, with an infinite T. If, on the other hand, the records are wiped out sufficiently frequently so that the threat of the punishment, in turn, is not too severe, the set of allocations that can be sustained is constrained, but it always includes some non-autarkic allocations. Figure 2

depicts $\Gamma_T(\beta)$ for any β and T positive and finite.

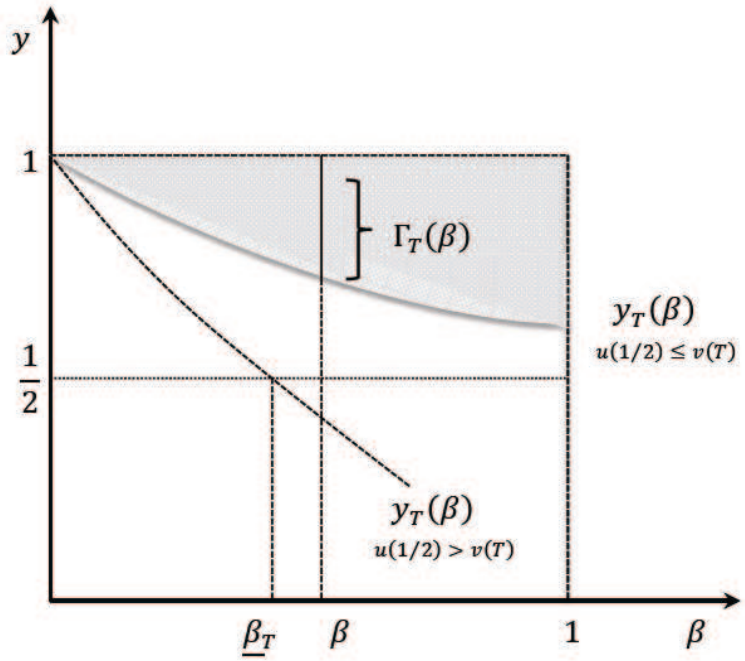

Figure 2: Sustainable allocations, $T > 0$ and finite

The following Lemma completes the characterization of the sustainable allocations for any given T, showing that they have some desirable properties which will turn out to be useful later on.

Lemma 11 *$\Gamma_T(\beta)$ is non-empty, compact, convex-valued and continuous in $\beta \in (0,1)$ for any $T \geq 0$.*

Proof. With $T = 0$, by Lemma 7, $\Gamma_0(\beta) = \{1\}$, hence, in this case the statement follows immediately. Consider $T > 0$. By Lemmas 9-10, for any $\beta \in (0,1)$ and $T > 0$, $\Gamma_T(\beta) = [\widehat{y}_T(\beta), 1]$ is a non-empty, closed and bounded interval of the real line, hence, $\Gamma_T(\beta)$ is non-empty, compact and convex-valued. For any $T > 0$, the upper boundary of $\Gamma_T(\beta)$ is constant and the lower boundary, $\widehat{y}_T(\beta)$, varies continuously with β, by the previous Lemmas 9-10, hence the correspondence $\Gamma_T(\beta)$ is continuous in β. ■

It is interesting to notice, as an aside, that the set of sustainable allocations becomes larger for larger values of T. This is intuitive, since the punishment associated with a deviation becomes more severe when trade is shut down for a longer period. Define, for given T, $Gr(\Gamma_T) \equiv \left\{(y,\beta) \in \left[\frac{1}{2},1\right] \times (0,1) \mid y \in \Gamma_T(\beta)\right\}$, the graph of the correspondence Γ_T.

Lemma 12 *$Gr(\Gamma_T) \subset Gr(\Gamma_{T'}) \subset Gr(\Gamma_\infty)$, for any finite T', T with $T' > T \geq 0$.*

Proof. The participation constraint can be rewritten as $\Psi_T(y,\beta) =$

$$g_T(\beta)\left\{\frac{f_T(\beta)}{g_T(\beta)}\left[u(1-y)-u(0)\right]-\left[u(1)-u(y)\right]\right\} \geq 0. \tag{13}$$

The term $g_T(\beta) = \frac{1-\beta^{T+2}}{1-\beta^2}$ is clearly increasing in T. The term $\frac{f_T(\beta)}{g_T(\beta)} = \beta\left(\frac{1-\beta^T}{1-\beta^{T+2}}\right) \leq \beta$, and approaches β when $T \to \infty$. Moreover, for any $\beta \in (0,1)$ and any T', T such that $T' > T \geq 0$, $\frac{f_T(\beta)}{g_T(\beta)} < \frac{f_{T'}(\beta)}{g_{T'}(\beta)}$, since $\beta\left(\frac{1-\beta^T}{1-\beta^{T+2}}\right) < \beta\left(\frac{1-\beta^{T'}}{1-\beta^{T'+2}}\right) \Leftrightarrow \beta^T\left(1-\beta^2\right)\left(1-\beta^{T'-T}\right) > 0$. Hence, the LHS of (13) is strictly higher for larger T, for any given β and y. ■

Finally, we show that, by concentrating on sustainable allocations, we are not leaving out any relevant allocation. Indeed, the allocations we have identified - including the autarkic ones- constitute all the (stationary) equilibria of the non-monetary regime.

Lemma 13 *An allocation y is a non-monetary Equilibrium if and only if $y \in \Gamma_T(\beta)$.*

Proof. i. "if" part. It follows from the definition of $\Gamma_T(\beta)$. ii. "only if" part. Suppose $y \notin \Gamma_T(\beta)$. Such an allocation would violate the participation constraint, hence, it could not be sustained as an equilibrium. ■

Welfare After having characterized all the sustainable allocations, we turn to the choice of the allocations that maximize ex-ante welfare among the ones that can be sustained. The allocation $y \in \Gamma_T(\beta)$ is chosen to maximize

$$W(y,\mu,\beta) = \frac{1}{1-\beta^2}\left\{\mu\left[u(y)+\beta u(1-y)\right]+(1-\mu)\left[u(1-y)+\beta u(y)\right]\right\}. \tag{14}$$

The Theorem of the Maximum implies that a solution exists and is well behaved for all feasible μ and β, for any T, as the next Lemma shows. Define $y_T^*(\mu,\beta) \equiv \arg\max\{W(y,\mu,\beta) \mid y \in \Gamma_T(\beta)\}$ and $W_T^*(\mu,\beta) \equiv \max\{W(y,\mu,\beta) \mid y \in \Gamma_T(\beta)\}$.

Lemma 14 *a. $y_T^*(\mu,\beta)$ is a single-valued, continuous function of μ and β for any T; b. $W_T^*(\mu,\beta)$ is a single-valued, continuous function of μ and β for any T.*

Proof. The function $W(y,\mu,\beta)$ is continuous in y, μ and β, strictly concave in y, and independent of T. This and Lemma 11 together ensure that Berge's (1997) Theorem of the Maximum applies. The statement follows. ■

5 Monetary Regime

The Clearing House We now consider a trading system that uses cash. The monetary facility works as follows. At date 0, an amount M_0 of divisible fiat money is available equally to agents of type 2. At each date t, the facility works in three stages: 1. agents can deliver one unit of the good in return for $\frac{1}{v_t}$ units of money and may receive a lump-sum transfer of money $\eta_t \in \mathbb{R}_+$ conditional on whether there was a delivery; 2. the facility collects lump-sum transfers of money, $\tau_t \in \mathbb{R}$, equally from (to) all agents and, in the case of negative transfers, destroys the corresponding amount; 3. agents can obtain v_t units of the good for every unit of money inserted in the facility. In the case of negative transfers, if some agent does not deliver the required amount, the facility stops operating. The facility can resume its functions following a wipe-out of the records after T periods, with $T = 2n$, $n \in \mathbb{N} \cup \{0\}$, exactly as before. Once re-started, the facility distributes a new currency and operates as before. The old currency is no longer accepted by the facility.[5] The lump-sum transfers, η_t and τ_t, are expressed in real terms, i.e. in consumption units, and are chosen to maximize ex-ante welfare. The workings of the facility, including v_t, η_t and τ_t at all t, are

[5] This assumption simplifies the analysis excluding the possibility of spending the old currency after the facility re-starts.

common knowledge at date 0. In the main body of the paper, we deal only with the case that does not discriminate between the types, i.e. in which $\eta_t = 0$ for all t. In the Appendix, we analyze the discriminatory[6] case in which $\eta_t > 0$ for an agent who delivers something at time t. Such a scheme works despite the presence of anonymity, since the transfers η_t are obtained by an agent only if he makes a delivery in the current period, and only agents with an endowment in the current period can deliver anything to the facility. The discriminatory scheme, which -whenever feasible- improves upon the non-discriminatory one, exploits the extreme nature of the endowment process, and may not work in more general environments.

In the monetary regime, the agents are allowed to optimize against their budget constraints, while deciding whether to participate in the transfer scheme. We will characterize, first, the allocations that satisfy the optimality and market clearing conditions, provisionally ignoring the participation decision. Then, we will consider the allocations that satisfy the participation conditions. We will call the allocations that satisfy optimality and participation, sustainable. Finally, we will choose the sustainable allocations that maximize ex-ante welfare.

Optimality and Market Clearing The maximization problem of agent i is

$$\underset{\left\{x_t^i\right\}_{t=1}^{\infty},\left\{m_t^i\right\}_{t=1}^{\infty}}{Max} \sum_{t=1}^{\infty} \beta^{t-1} u\left(x_t^i\right),$$

$$s.t. \quad x_t^i + v_t m_t^i = e_t^i + v_t m_{t-1}^i + \tau_t, \ \forall t \geq 1, \tag{15}$$

$$m_t^i \geq 0, \forall t \geq 1, \tag{16}$$

where m_t^i is the amount of money owned by agent i in period t. Notice that lump-sum transfers are not indexed by the agent's type. This is coherent with the assumption that the agents are anonymous and indistinguishable from the point of view of the facility that permits trade. Given the Inada condition on the utility function we

[6]The term "discriminatory", referred to this type of transfers, is borrowed from Sargent (1987), chapter 6.

do not need to worry about the non-negativity constraint on consumption. Market clearing for the good requires

$$Nx_t^1 + Nx_t^2 = N, \forall t \geq 1, \tag{17}$$

while market clearing for money is implied by Walras Law. We consider, first, the allocations that solve the maximization problem above and satisfy (17), ignoring for the moment the participation constraint.

The first order conditions for an optimum are

$$\beta^{t-1} u'\left(x_t^i\right) = \lambda_t^i, \forall t \geq 1, \tag{18}$$

for the consumption choice, where $\lambda_t^i > 0, \forall t \geq 1$, is the multiplier of the constraint in (15), and

$$-v_t \lambda_t^i + v_{t+1} \lambda_{t+1}^i + \theta_t^i = 0, \forall t \geq 1, \tag{19}$$

for the choice of money holdings, where $\theta_t^i \geq 0, \forall t \geq 1$, is the multiplier of the non-negativity constraint on money holdings, (16). There is also the complementary slackness condition for the non-negativity constraint on money holdings, (16),

$$\theta_t^i m_t^i = 0, \forall t \geq 1. \tag{20}$$

We consider the following candidate for a solution of the maximization problem: $m_t^1 = 0$ when $t = 2n, n \in \mathbb{N}$, $m_t^1 > 0$ when $t = 2n-1, n \in \mathbb{N}$, and viceversa for agents of type 2. In other words, the candidate solution requires the agents to demand a positive amount of money when they receive their endowment of the good, and spend entirely their money holdings before receiving any new endowment. In this situation, the equations (18) and (19) give

$$u'\left(x_t^i\right) = \beta \frac{v_{t+1}}{v_t} u'\left(x_{t+1}^i\right), \tag{21}$$

for $i = 1, t = 2n-1, n \in \mathbb{N}$ and $i = 2, t = 2n, n \in \mathbb{N}$. In the other cases $u'\left(x_t^i\right) \geq \beta \frac{v_{t+1}}{v_t} u'\left(x_{t+1}^i\right)$.

The stock of money in any period t is given by $M_t = M_{t-1} + \frac{2N\tau_t}{v_t}$. Define $\pi_t \equiv \frac{2N\tau_t}{v_t M_{t-1}}$, for all $t \geq 1$, thus, $M_t = (1+\pi_t) M_{t-1}$, for all $t \geq 1$. As in the case of the non-monetary facility we look at symmetric and stationary allocations. Hence, we look at situations in which $\pi_t = \pi \in [\beta - 1, \infty)$, for all $t \geq 1$.[7] Stationarity implies also that $v_t (1+\pi) = v_{t-1}$ for all $t \geq 1$. There are always circumstances in which cash is not valued, i.e. $v_t = 0$ for all $t \geq 1$, and, therefore, agents do not trade. Henceforth, we concentrate on the case in which cash has value at all times and some trade can occur.

The stationary allocation of consumption that satisfies market clearing, (17), is cyclical of order two and entails for an agent of type 1 x units of consumption in odd periods and $1-x$ in even periods and viceversa for type 2 agents. The allocations must satisfy

$$\Phi(x, \pi, \beta) \equiv u'(x) - \frac{\beta}{1+\pi} u'(1-x) = 0. \tag{22}$$

The next Lemma establishes that, for any admissible inflation or deflation rate and discount rate, a unique non-autarkic allocation exists that satisfies (22).

Lemma 15 *An allocation $\widetilde{x} \in \left[\frac{1}{2}, 1\right)$ that solves (22) exists and is unique for every $\pi \in [\beta - 1, \infty)$ and $\beta \in (0,1)$.*

Proof. $\Phi(x, \pi, \beta)$ is (at least once) continuously differentiable in x, with $\Phi(1, \pi, \beta) = -\infty$, and $\Phi\left(\frac{1}{2}, \pi, \beta\right) = u'(x)\left(1 - \frac{\beta}{1+\pi}\right) \geq 0$. Hence, by the Intermediate Value Theorem, there exists a value $\widetilde{x} \in \left[\frac{1}{2}, 1\right)$ that solves (22) for any $\pi \in [\beta - 1, \infty)$ and $\beta \in (0,1)$. Moreover, $\widetilde{x}$ is unique for any π and β, since $\frac{\partial \Phi(x,\pi,\beta)}{\partial x} = u''(x) + \frac{\beta}{1+\pi} u''(1-x) < 0$ for all $\pi \in [\beta - 1, \infty)$, $\beta \in (0,1)$ and $x \in \left[\frac{1}{2}, 1\right)$. ■

Define $\widetilde{x}(\pi, \beta) : [\beta - 1, \infty) \times (0,1) \to \left[\frac{1}{2}, 1\right]$ as the (at least once) continuously differentiable function, $\widetilde{x} = \widetilde{x}(\pi, \beta)$, such that the values $(\widetilde{x}, \pi, \beta)$ satisfy $\Phi(\widetilde{x}, \pi, \beta) = 0$. The next Lemma characterizes the behavior of the solutions as the monetary policy parameter varies over its feasible range.

[7] We exclude $\pi < \beta - 1$, since it would be at odds with the existence of a monetary equilibrium.

Lemma 16 *a. The function* $\widetilde{x}(\pi,\beta)$ *is at least once continuously differentiable in* π*;*

b. i. $\widetilde{x}(\beta-1,\beta)=\frac{1}{2}$*, ii.* $\lim_{\pi\to\infty}\widetilde{x}(\pi,\beta)=1$ *for any* $\beta\in(0,1)$*;*

c. the derivative $\frac{\partial\widetilde{x}(\pi,\beta)}{\partial\pi}>0$ *for any* $\beta\in(0,1)$*.*

Proof. Part a. and part c., follow from the Implicit Function Theorem, since $\frac{\partial\Phi(x,\pi,\beta)}{\partial x}\|_{x=\widetilde{x}}<0$ from the previous Lemma and $\frac{\partial\Phi(x,\pi,\beta)}{\partial\pi}\|_{x=\widetilde{x}}=\frac{\beta}{(1+\pi)^2}u'(1-\widetilde{x})>0$. Part b.i. is obvious from inspection of (22) and b. ii. from the Inada condition. ■

Define the correspondence $\widetilde{\Gamma}(\pi,\beta):[\beta-1,\infty)\times(0,1)\rightrightarrows\left[\frac{1}{2},1\right]$ as $\widetilde{\Gamma}(\pi,\beta)\equiv[\widetilde{x}(\pi,\beta),1]$.

Lemma 17 $\widetilde{\Gamma}(\pi,\beta)\subset\widetilde{\Gamma}(\pi',\beta)$*, for any* $\pi,\pi'\in[\beta-1,\infty)$*, with* $\pi'<\pi$*, for any* $\beta\in(0,1)$*.*

Proof. By definition $\widetilde{\Gamma}(\pi,\beta)\equiv[\widetilde{x}(\pi,\beta),1]$. By Lemma 16, part c, $\frac{\partial\widetilde{x}(\pi,\beta)}{\partial\pi}>0$ for any $\beta\in(0,1)$. The statement follows. ■

Participation Consider now the decision of an agent whether to pay the taxes or not. This decision is relevant only if $\tau<0$, i.e. $\pi\in[\beta-1,0)$. If an agent decides to stick to the taxation regime operated by the monetary facility, will get the implied allocation $\widetilde{x}=\widetilde{x}(\pi,\beta)$, while if it decides not to pay the taxes the facility will stop operating for T periods, shutting down trade for all the agents. Consider an agent who is receiving an endowment in the current period. If he decides not to pay taxes in the current period, he anticipates that the facility will stop for some time and will eventually resume its activity but with a new currency. Hence, it is pointless, for an agent who has decided not to pay taxes, to deliver anything to the facility. Therefore, for such an agent the taxation τ and the implied deflation rate π will have to satisfy the following participation constraint

$$g_T(\beta)\,u(\widetilde{x})+f_T(\beta)\,u(1-\widetilde{x})\geq g_T(\beta)\,u(1)+f_T(\beta)\,u(0)\,, \tag{23}$$

where $f_T(\beta)$ and $g_T(\beta)$ have been defined before. Consider an agent who does not receive an endowment in the current period. If he decides not to pay taxes, the consequence for him will be the impossibility to consume in the current period, followed by autarky for T periods. Hence, for this type of agent, the participation constraint is given by

$$g_T(\beta)\, u(1-\widetilde{x}) + f_T(\beta)\, u(\widetilde{x}) \geq f_T(\beta)\, u(1) + g_T(\beta)\, u(0). \tag{24}$$

Whenever an allocation satisfies (23), also satisfies (24), as the next Lemma shows.

Lemma 18 *For any $\beta \in (0,1)$ and $T \geq 0$, if $\widetilde{x}$ satisfies (23), then it satisfies (24).*

Proof. Rearrange (23) and (24) to obtain, respectively, $f_T(\beta)\left[u(1-\widetilde{x}) - u(0)\right] \geq g_T(\beta)\left[u(1) - u(\widetilde{x})\right]$ and $g_T(\beta)\left[u(1-\widetilde{x}) - u(0)\right] \geq f_T(\beta)\left[u(1) - u(\widetilde{x})\right]$. Notice that $g_T(\beta) = \frac{1-\beta^{T+2}}{1-\beta^2} > \beta\frac{1-\beta^T}{1-\beta^2} = f_T(\beta)$ for any $\beta \in (0,1)$ and $T \geq 0$. The statement follows. ■

Therefore, to ensure agents' participation, we can ignore the latter constraint and work only with the former. The constraint (23) places a lower bound on the deflation rates that can be achieved in a monetary economy and is the same as the participation condition (8) of the non-monetary economy. In the monetary economy, however, it applies only in the case of taxes, i.e. for a deflation. Since the constraint is the same as before, we keep using $\Gamma_T(\beta)$ to denote the allocations that satisfy the participation constraint.

Sustainable Allocations We call sustainable, the allocations that satisfy optimality and the participation constraint.

Definition 2 *An allocation that satisfies (22) for any $\pi \in [\beta - 1, \infty)$ and $\beta \in (0,1)$, and (23) when $\pi \in [\beta - 1, 0)$, is sustainable with cash.*

Define the correspondence $\Gamma_T^M(\beta) : (0,1) \rightrightarrows \left[\frac{1}{2}, 1\right]$ as

$$\Gamma_T^M(\beta) \equiv \left(\left(\widetilde{\Gamma}(\beta - 1, \beta) \setminus \widetilde{\Gamma}(0,\beta)\right) \cap \Gamma_T(\beta)\right) \cup \widetilde{\Gamma}(0,\beta), \tag{25}$$

mapping values of β into allocations that are sustainable as monetary equilibria.

Lemma 19 *An allocation x is sustainable with cash if and only if $x \in \Gamma_T^M(\beta)$.*

Proof. i. "if" part. The set $\left(\widetilde{\Gamma}(\beta-1,\beta) \setminus \widetilde{\Gamma}(0,\beta)\right)$ represents the allocations that satisfy (22) for $\pi \in [\beta-1,0)$.[8] These allocations are constrained by $\Gamma_T(\beta)$, the set of allocations that satisfy participation (23). Finally, $\widetilde{\Gamma}(0,\beta)$ represents the allocations that satisfy (22) for any $\pi \in [0,\infty)$, which are unconstrained. Hence, if $x \in \Gamma_T^M(\beta)$, it is sustainable according to Definition 2. ii. "only if" part. Suppose $x \notin \Gamma_T^M(\beta)$. Then, x violates either (22) for $\pi \in [\beta-1,\infty)$ or (23) for $\pi \in [\beta-1,0)$, or both, hence, it is not sustainable according to Definition 2. ■

The next Lemma provides a complete characterization of sustainable allocations.

Lemma 20 *$\Gamma_T^M(\beta)$ is non-empty, compact, convex-valued and continuous in $\beta \in (0,1)$ for any $T \geq 0$.*

Proof. The set $\widetilde{\Gamma}(0,\beta) = [\widetilde{x}(0,\beta),1]$ is non-empty, since $\widetilde{x}(0,\beta) < 1$ for any $\beta \in (0,1)$, compact, convex-valued and continuous in β since $\widetilde{x}(0,\beta)$ is continuous in β by Lemma 16. The set $\left(\widetilde{\Gamma}(\beta-1,\beta) \setminus \widetilde{\Gamma}(0,\beta)\right) = \left[\frac{1}{2},1\right] \setminus [\widetilde{x}(0,\beta),1] = \left[\frac{1}{2},\widetilde{x}(0,\beta)\right)$ is non-empty, since $\widetilde{x}(0,\beta) > \frac{1}{2}$, by (22) with $\pi = 0$, for any $\beta \in (0,1)$. The set $\Gamma_T(\beta) = [\widehat{y}_T(\beta),1]$ is non-empty, compact, convex-valued and continuous in β for any $T \geq 0$ by Lemma 11. The set $\left(\widetilde{\Gamma}(\beta-1,\beta) \setminus \widetilde{\Gamma}(0,\beta)\right) \cap \Gamma_T(\beta) = \left[\frac{1}{2},\widetilde{x}(0,\beta)\right) \cap [\widehat{y}_T(\beta),1]$ could be: 1. empty, if $\widehat{y}_T(\beta) \geq \widetilde{x}(0,\beta)$; or 2. equal to $[\widehat{y}_T(\beta),\widetilde{x}(0,\beta))$, if $\widehat{y}_T(\beta) < \widetilde{x}(0,\beta)$. The set $\Gamma_T^M(\beta) = \left[\frac{1}{2},\widetilde{x}(0,\beta)\right) \cap [\widehat{y}_T(\beta),1] \cup [\widetilde{x}(0,\beta),1]$ is equal to $[\widetilde{x}(0,\beta),1]$ in case 1. and $[\widehat{y}_T(\beta),\widetilde{x}(0,\beta)) \cup [\widetilde{x}(0,\beta),1] = [\widehat{y}_T(\beta),1]$ in case 2. In either case, $\Gamma_T^M(\beta)$ is non-empty, compact, convex-valued and continuous in β for any $T \geq 0$. ■

The next Lemma shows that the allocations we have characterized constitute all the (stationary) monetary equilibria of our economy.

[8] By Lemma 17, $\widetilde{\Gamma}(0,\beta) \subset \widetilde{\Gamma}(\beta-1,\beta)$, for any β. Thus, $\widetilde{\Gamma}(\beta-1,\beta) \setminus \widetilde{\Gamma}(0,\beta)$ is the complement of $\widetilde{\Gamma}(0,\beta)$ in $\widetilde{\Gamma}(\beta-1,\beta)$ for any β.

Lemma 21 *An allocation x is a Monetary Equilibrium, if and only if $x \in \Gamma_T^M(\beta)$.*

Proof. i. "if" part. The bounded sequence of consumption $(x, 1-x)$ and money holdings $(m, 0)$ repeating itself identically every other period for agents of type 1 and viceversa for agents of type 2, satisfies market clearing and the necessary condition for an optimum. It also satisfies the transversality condition, $\lim_{t\to\infty} -\beta^{t-1}u'(x_t^i)\, v_t m_t^i = 0$. Thus, it constitutes an unconstrained monetary equilibrium for every π. The values of $\pi \in [\beta - 1, 0)$ which would imply non participation in the lump-sum taxation scheme are excluded by the imposition of (23). ii. "only if" part. An allocation $x \notin \Gamma_T^M(\beta)$ would violate either the necessary condition for an optimum or the participation constraint or both, hence it cannot be a monetary equilibrium. ■

Welfare We move to the maximization of the ex-ante welfare. We know that monetary equilibrium allocations and inflation rates, are related by a function $\widetilde{x}(\pi, \beta)$ which is strictly increasing. Although it would be natural, economically, to think of π as the variable chosen to maximize ex-ante welfare, it is equivalent and more convenient for the purpose of the comparison with the non-monetary trading system, to let the allocation be the choice variable. Since $\widetilde{x}(\pi, \beta)$ is invertible one can always derive the implied inflation or deflation rate. Therefore, $x \in \Gamma_T^M(\beta)$ is chosen to maximize

$$W^M(x, \mu, \beta) = \frac{1}{1-\beta^2}\left\{\mu\left[u(x) + \beta u(1-x)\right] + (1-\mu)\left[u(1-x) + \beta u(x)\right]\right\}. \quad (26)$$

The Theorem of the Maximum implies that a solution exists and is well behaved for all feasible μ and β, for any T, as the next Lemma shows. Define $x_T^{M*}(\mu, \beta) \equiv \arg\max\left\{(26) \mid x \in \Gamma_T^M(\beta)\right\}$ and $W_T^{M*}(\mu, \beta) \equiv \max\left\{(26) \mid x \in \Gamma_T^M(\beta)\right\}$.

Lemma 22 *a. $x_T^{M*}(\mu, \beta)$ is a single-valued, continuous function of μ and β for any T; b. $W_T^{M*}(\mu, \beta)$ is a single-valued, continuous function of μ and β for any T.*

Proof. The function $W^M(x, \mu, \beta)$ is continuous in x, μ and β, strictly concave in x, and independent of T. This and Lemma 20 together ensure that Berge's (1997) Theorem of the Maximum applies. The statement follows. ■

6 Comparison of the Regimes

Set Inclusion We begin the comparison of the non-monetary and monetary regimes with the observation that the set of equilibrium allocations obtained under the monetary regime cannot be smaller than the one obtained under the non-monetary regime.

Proposition 1 *$\Gamma_T(\beta) \subseteq \Gamma_T^M(\beta)$, for any $\beta \in (0,1)$ and $T \geq 0$.*

Proof. $\Gamma_T^M(\beta) \equiv \left(\left(\widetilde{\Gamma}(\beta-1,\beta) \setminus \widetilde{\Gamma}(0,\beta)\right) \cap \Gamma_T(\beta)\right) \cup \widetilde{\Gamma}(0,\beta)$ by definition. For any given $\beta \in (0,1)$ and $T \geq 0$, there are two possible cases: the intersection is empty or not. 1. $\left(\widetilde{\Gamma}(\beta-1,\beta) \setminus \widetilde{\Gamma}(0,\beta)\right) \cap \Gamma_T(\beta) = \varnothing$. Since $\widetilde{\Gamma}(\beta-1,\beta) \setminus \widetilde{\Gamma}(0,\beta) = \left[\frac{1}{2},1\right] \setminus [\widetilde{x}(0,\beta),1] = \left[\frac{1}{2},\widetilde{x}(0,\beta)\right)$ and $\Gamma_T(\beta) = [\widehat{y}_T(\beta),1]$ for the intersection to be empty it must be the case that $\widehat{y}_T(\beta) \geq \widetilde{x}(0,\beta)$, therefore $\Gamma_T^M(\beta) = (\varnothing \cup [\widetilde{x}(0,\beta),1]) = [\widetilde{x}(0,\beta),1] \supseteq [\widehat{y}_T(\beta),1] = \Gamma_T(\beta)$. Clearly, the inclusion is strict if $\widehat{y}_T(\beta) > \widetilde{x}(0,\beta)$, while the two sets coincide if $\widehat{y}_T(\beta) = \widetilde{x}(0,\beta)$. 2. $\left(\widetilde{\Gamma}(\beta-1,\beta) \setminus \widetilde{\Gamma}(0,\beta)\right) \cap \Gamma_T(\beta) \neq \varnothing$. For the intersection to be non-empty it must be the case that $\widehat{y}_T(\beta) < \widetilde{x}(0,\beta)$, therefore $\Gamma_T^M(\beta) = \left(\left[\frac{1}{2},\widetilde{x}(0,\beta)\right) \cap [\widehat{y}_T(\beta),1]\right) \cup [\widetilde{x}(0,\beta),1] = [\widehat{y}_T(\beta),\widetilde{x}(0,\beta)) \cup [\widetilde{x}(0,\beta),1] = [\widehat{y}_T(\beta),1] = \Gamma_T(\beta)$. ■

Although both systems have to ensure that agents have the incentive to deliver some resources, they do so in quite different ways. The cash-less system works thanks to the threat of collective punishment, while the cash-based system only needs such a threat to induce agents to pay taxes. The participation constraint is identical in the two regimes, but in the monetary system it applies to a more limited set of circumstances. Thus, the cash-based system can always sustain at least the same allocations as the cash-less system. We will see below that there are robust cases in which it can sustain strictly more. Before doing that, we turn to the welfare comparison of the two regimes.

Welfare Comparison Since the set of allocations that constitute a Monetary Equilibrium is never smaller than the set of allocations that constitute a non-monetary

Equilibrium and the welfare function is the same in the two cases, the maximized welfare in the Monetary Equilibrium cannot be smaller than in the non-monetary Equilibrium.

Proposition 2 *$W_T^*(\mu,\beta) \leq W_T^{M*}(\mu,\beta)$, for any $\mu \in [0,1]$, $\beta \in (0,1)$ and $T \geq 0$.*

Proof. The objective functions (14) and (26) are identical. By Proposition 1, $\Gamma_T(\beta) \subseteq \Gamma_T^M(\beta)$ for any β and T. The statement follows by definition of $W_T^*(\mu,\beta)$ and $W_T^{M*}(\mu,\beta)$. ■

The ex-ante welfare functions in the non-monetary and monetary regimes are the same, given by

$$\frac{1}{1-\beta^2}\left\{\mu\left[u(h)+\beta u(1-h)\right]+(1-\mu)\left[u(1-h)+\beta u(h)\right]\right\}. \tag{27}$$

with $h \in \mathbb{R}_+$. Consider the problem of maximizing the ex-ante welfare function with the only constraint that the choice should be feasible, i.e. maximize (27) in $h \in \left[\frac{1}{2},1\right]$. The objective function (27) is (at least twice) continuously differentiable, strictly increasing and strictly concave in the choice variable, h. Hence, for any (μ,β) there exists a unique, global maximizer, which is characterized by the following necessary and sufficient conditions

$$\mu\left[u'(h)-\beta u'(1-h)\right]+(1-\mu)\left[-u'(1-h)+\beta u'(h)\right]-\rho+\nu=0, \tag{28}$$

$$\rho(1-h)=0, \tag{29}$$

$$\nu\left(h-\frac{1}{2}\right)=0, \tag{30}$$

where $\rho \geq 0$ and $\nu \geq 0$ are the multipliers for the boundary conditions on h. Define $h^*(\mu,\beta)$:$[0,1]\times(0,1) \to \left[\frac{1}{2},1\right]$ as the function that satisfies (28), (29), (30). Define also $\widetilde{h}(\beta)$:$(0,1) \to \left[\frac{1}{2},1\right]$ as the function that satisfies

$$u'(h)-\beta u'(1-h)=0, \tag{31}$$

for any $\beta \in (0,1)$. Such a function is continuous in $\beta \in (0,1)$, by the same argument used in Lemma 16 with $\pi=0$.

Proposition 3 *$h^*(\mu,\beta) \leq \widetilde{h}(\beta)$ for all $\mu \in [0,1]$ at any $\beta \in (0,1)$.*

Proof. First, observe that $\rho\nu = 0$. Second, $\rho = 0$ always. Suppose $\rho > 0$, instead. By (29), $h = 1$ and (28) gives $\rho = -\infty$, which contradicts $\rho > 0$. Define

$$\Phi(h,\mu,\beta) \equiv \mu\left[u'(h) - \beta u'(1-h)\right] + (1-\mu)\left[-u'(1-h) + \beta u'(h)\right] + \nu = 0,$$

where $\Phi(1,\mu,\beta) = -\infty$, $\Phi\left(\frac{1}{2},\mu,\beta\right) = (1-\beta)\,u'\left(\frac{1}{2}\right)(2\mu-1) + \nu$, and

$$\frac{\partial\Phi(h,\mu,\beta)}{\partial h} = u''(h)(\mu+\beta-\mu\beta) + u''(1-h)(1-\mu+\mu\beta) < 0.$$

Hence, for $\mu \in \left[0,\frac{1}{2}\right)$, $\nu > 0$ and $h^*(\mu,\beta) = \frac{1}{2} < \widetilde{h}(\beta)$ by (30) and (31) for any $\beta \in (0,1)$. For $\mu \in \left[\frac{1}{2},1\right]$, we have $\nu = 0$. Observe that $\Phi(h,1,\beta) = u'(h) - \beta u'(1-h) = 0$, which gives $h^*(1,\beta) = \widetilde{h}(\beta)$ for any $\beta \in (0,1)$, and $\Phi\left(h,\frac{1}{2},\beta\right) = \frac{1}{2}(1+\beta)\left[u'(h) - u'(1-h)\right] = 0$, which gives $h^*\left(\frac{1}{2},\beta\right) = \frac{1}{2} < \widetilde{h}(\beta)$ for any $\beta \in (0,1)$. The derivative

$$\frac{\partial h^*(\mu,\beta)}{\partial\mu} = -\frac{(1-\beta)\left[u'(h) + u'(1-h)\right]}{u''(h)(\mu+\beta-\mu\beta) + u''(1-h)(1-\mu+\mu\beta)} > 0.$$

The statement follows. ■

The consequence of Propositions 1 and 3, is that if the set of non-monetary equilibria is strictly included in the set of monetary equilibria, welfare is strictly higher in the latter than in the former for any μ, and viceversa, as the next Proposition shows.

Proposition 4 *For any $\mu \in [0,1]$, $W_T^*(\mu,\beta) < W_T^{M*}(\mu,\beta)$, if and only if $\Gamma_T(\beta) \subset \Gamma_T^M(\beta)$, for some $\beta \in (0,1)$ and $T \geq 0$.*

Proof. i. "if" part. From the Proof of Proposition 1, $\Gamma_T(\beta) \subset \Gamma_T^M(\beta) \Leftrightarrow \widehat{y}_T(\beta) > \widetilde{x}(0,\beta)$ for any given $\beta \in (0,1)$ and $T \geq 0$. From Proposition 3, $h^*(\mu,\beta) \leq \widetilde{h}(\beta)$ for all $\mu \in [0,1]$ at any given $\beta \in (0,1)$. By definition, $\widetilde{x}(0,\beta) \equiv \widetilde{h}(\beta)$ for any given $\beta \in (0,1)$. If $\widehat{y}_T(\beta) > \widetilde{x}(0,\beta)$ for some $\beta \in (0,1)$ and $T \geq 0$, we have $h^*(\mu,\beta) \leq \widetilde{h}(\beta) = \widetilde{x}(0,\beta) < \widehat{y}_T(\beta)$, for any given $\mu \in [0,1]$, at those values of $\beta \in (0,1)$ and $T \geq 0$. Since (27) is strictly concave in h and $h^*(\mu,\beta)$ is the global maximum for any given $\mu \in [0,1]$

and $\beta \in (0,1)$, the function (27) is strictly decreasing in h for any $h > h^*(\mu,\beta)$, for given $\mu \in [0,1]$ and $\beta \in (0,1)$. By definition, $W_T^*(\mu,\beta) = \max\{(27) \mid h \in [\widehat{y}_T(\beta),1]\}$ and $W_T^{M*}(\mu,\beta) = \max\{(27) \mid h \in [\widetilde{x}(0,\beta),1]\}$. The statement follows. ii. "only if" part. Suppose, $\Gamma_T(\beta) = \Gamma_T^M(\beta)$ for some $\beta \in (0,1)$ and $T \geq 0$. The objective functions (14) and (26) are identical. The statement follows by definition of $W_T^*(\mu,\beta)$ and $W_T^{M*}(\mu,\beta)$. ■

The reader may now wonder whether the strict inclusion holds generally. The answer is negative. Indeed, one can find examples of economies in which it is never true for any β. Below, we provide first a necessary, then a sufficient condition for the strict inclusion to hold.

Finite T is Necessary The next proposition shows that $T < \infty$ is necessary to have the strict inclusion.

Proposition 5 *$\Gamma_\infty(\beta) = \Gamma_\infty^M(\beta)$ for any $\beta \in (0,1)$.*

Proof. For any β, $\widetilde{x}(0,\beta)$ satisfies $\beta = f(\widetilde{x}) \equiv \frac{u'(\widetilde{x})}{u'(1-\widetilde{x})}$ and for $T = \infty$ the participation constraint is

$$\Psi_\infty(x,\beta) = \frac{1}{1-\beta^2}\left[\beta\left(u(1-x) - u(0)\right) - \left(u(1) - u(x)\right)\right] \geq 0$$

that holds true if and only if

$$\beta \geq \frac{u(1) - u(x)}{u(1-x) - u(0)} \tag{32}$$

In turn, we have that

$$\beta = f(\widetilde{x}) \equiv \frac{u'(\widetilde{x})}{u'(1-\widetilde{x})} > \frac{u(1) - u(\widetilde{x})}{u(1-\widetilde{x}) - u(0)},$$

where the inequality holds for any $\widetilde{x}$ by strict concavity of the utility function. Hence, (32) holds at $\widetilde{x}(0,\beta)$ for any $\beta \in (0,1)$. Therefore, for any $\beta \in (0,1)$, $\Gamma_\infty^M(\beta) = ([\widehat{y}_\infty(\beta),\widetilde{x}(0,\beta)) \cup [\widetilde{x}(0,\beta),1]) = [\widehat{y}_\infty(\beta),1] = \Gamma_\infty(\beta)$. ■

By Propositions 2, 4 and 5, we immediately have the following

Corollary 1 $W_{\infty}^{*}(\mu,\beta) = W_{\infty}^{M*}(\mu,\beta)$, *for any* $\mu \in [0,1]$ *and* $\beta \in (0,1)$.

Hence, for money to be strictly essential, it necessarily has to be that the record of past deviations is wiped out in finite time and, thus, the ability of the clearing house to credibly enforce the threat of punishing shortfalls in the contributions to the pool is less than perfect. This is only a necessary condition, though, and the reader may still wonder whether the strict inclusion ever really happens. The next section provides a sufficient condition under which the strict inclusion indeed occurs.

Strict Inclusion and Essentiality Proposition 6 provides a sufficient condition for the existence of an interval of values of β such that the inclusion is, indeed, strict, in economies with a finite T . In Section 4, we defined $\overline{T} \equiv \frac{2\left[u(1)-u\left(\frac{1}{2}\right)\right]}{2u\left(\frac{1}{2}\right)-u(1)-u(0)} \in (0,\infty)$.

Proposition 6 *If* $T < \overline{T}$, *there exists an interval* $\Delta_T \subseteq (0,1)$ *with non-empty interior such that* $\Gamma_T(\beta) \subset \Gamma_T^M(\beta)$ *if* $\beta \in \Delta_T$.

Proof. For any $T < \infty$, $\widehat{y}_T(\beta)$ and $\widetilde{x}(0,\beta)$ are continuous in $\beta \in (0,1)$, by Lemmas 9-10 and 16 respectively. When $T < \overline{T}$, we have $\lim_{\beta\to 1}\widehat{y}_T(\beta) = \overline{y}_T > \frac{1}{2}$; moreover, $\lim_{\beta\to 1}\widetilde{x}(0,\beta) = \frac{1}{2}$. Therefore, by continuity, there exists an interval $\Delta_T \subseteq (0,1)$ with non-empty interior, such that $\widehat{y}_T(\beta) > \widetilde{x}(0,\beta)$, for $\beta \in \Delta_T$, and, thus, $\Gamma_T(\beta) = [\widehat{y}_T(\beta),1] \subset [\widetilde{x}(0,\beta),1] = \Gamma_T^M(\beta)$, for $\beta \in \Delta_T$. ■

Hence, if the records of past deviations is wiped out sufficiently frequently, and, thus, the enforcement power of the clearing house is sufficiently limited, the money-based trading regime achieves a strictly larger set of allocations. The strict inclusion occurs when the allocation that constitutes a monetary equilibrium without taxation cannot be reached under the cash-less trading system because the punishment is not sufficiently effective and, thus, the set of allocations that can be sustained without cash is very limited, while the cash-based economy is not subject to a participation constraint in the absence of taxation. The possibility of re-starting the facility in finite time following a complete wipe-out of the records, constitutes precisely a limit

to the effectiveness of punishment. Figure 3 depicts the strict inclusion case.

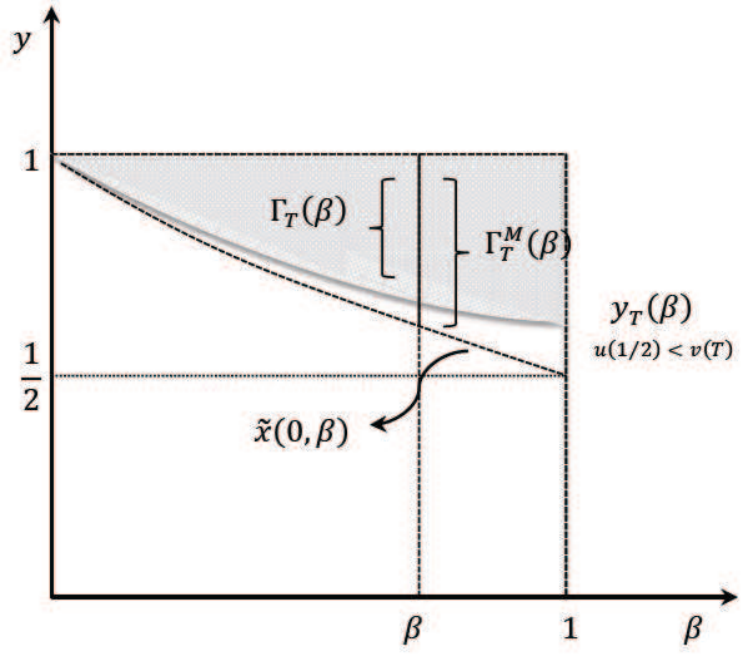

Figure 3: Comparison of the Regimes

It is now enough to gather our findings together to obtain the final step of our analysis. Proposition 4 and 6 together imply our main result.

Proposition 7 *Assume* $T < \overline{T}$. *For any* $\mu \in [0,1]$, *if* $\beta \in \Delta_T$, *then* $W_T^*(\mu,\beta) < W_T^{M*}(\mu,\beta)$.

Proof. Under the assumption, by Proposition 6, $\beta \in \Delta_T \Rightarrow \Gamma_T(\beta) \subset \Gamma_T^M(\beta)$. By Proposition 4, given β and T, $\Gamma_T(\beta) \subset \Gamma_T^M(\beta) \Leftrightarrow W_T^*(\mu,\beta) < W_T^{M*}(\mu,\beta)$, for any $\mu \in [0,1]$. The statement follows. ■

This completes our argument. The monetary trading system, even without discriminatory taxation, allows to achieve always at least the same allocations and the same welfare as the non-monetary one, and in some cases it allows to achieve a strictly larger set of allocations which are also strictly better from the point of view of the

agents' ex-ante welfare. The monetary system is more flexible than the non-monetary one, since only the allocations corresponding to deflationary price sequences, obtained through taxation, need to be induced via the threat of collective punishment, while all other allocations can be selected optimally by the agents themselves. When enforcement is sufficiently imperfect, such higher flexibility emerges fully and determines the superiority of the cash-based system.

7 Conclusion

We have proposed a model of intertemporal trade where exchange is conducted through a multilateral clearing house whose smooth working is hindered by the inability to keep the record of past deviations indefinitely and, thus, perfectly enforce punishments. We found that the clearing house, while being able to operate quite well even without making use of money, can always operate at least as effectively and sometimes strictly improve its functioning through the use of cash, provided the loss of the records occurs frequently enough. In future research, we intend to explore an environment with intratemporal trade.

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8 Appendix

Discriminatory Transfers We consider, in this Appendix, the case in which a positive transfer is given to an agent who delivers something, while nothing is given to agents who do not deliver anything. The term η_t now appears on the RHS of the budget constraint (15)

$$x_t^i + v_t m_t^i = e_t^i + v_t m_{t-1}^i + \eta_t + \tau_t, \ \forall t \geq 1.$$

The rest remains the same. Specifically, we consider the case in which $\eta_t = -\tau_t$ when $e_t^i > 0$, $\eta_t = 0$ when $e_t^i = 0$, and $\tau_t < 0$. Through this scheme, the monetary facility effectively sterilizes the taxation for one of the two types of agents every period, thus, implementing an asymmetric (or discriminatory) transfer scheme across types of agents. The scheme works despite the presence of anonymity, since the transfers η_t are conditional on a delivery being made in the current period, and only agents with an endowment in the current period can deliver anything to the facility. This would limit the applicability of such a scheme in case of a more general endowment process.

By fixing τ_t appropriately, any allocation $z(\mu)$ on the first best frontier - identified in Lemma 1- can be achieved, provided the agents are exogenously assumed to abide by the transfer scheme.

Lemma 23 *Any $z(\mu)$ is achieved in a Monetary Equilibrium with discriminatory transfers, choosing τ_t appropriately, when the agents' participation in the discriminatory transfer scheme is exogenously assumed.*

Proof. The proof of this statement mirrors exactly the proof of Proposition 2 in Townsend (1980), once we notice that the type specific transfers of Townsend (z_t^i, in his notation), can be replicated by our type independent transfers with $\eta_t = -\tau_t > 0$ or $\eta_t = 0$, conditional on having delivered some amount of the good to the facility at the delivery stage. ■

Since in our economy, agents cannot commit to stick to the proposed transfer scheme, we need to check which first best allocations satisfy the participation constraints. An agent who is in a position to make a delivery in the current period, might decide not pay taxes, in which case, given the way the facility works, it is a dominant strategy not to deliver anything. This gives rise to the following participation constraint

$$h_T(\beta)\, u(z) \geq g_T(\beta)\, u(1) + f_T(\beta)\, u(0), \tag{33}$$

where $h_T(\beta) \equiv \sum_{j=0}^{T} \beta^j$ and $f_T(\beta)$, $g_T(\beta)$ are defined as before. On the other hand, an agent who does not receive an endowment in the current period, will participate in the taxation scheme if

$$h_T(\beta)\, u(1-z) \geq f_T(\beta)\, u(1) + g_T(\beta)\, u(0). \tag{34}$$

Definition 3 *An allocation $z \in [0,1]$ satisfying (33) and (34) is sustainable as a Monetary Equilibrium with discriminatory transfers.*

As noted before, very little can be sustained when the threat of the punishment is most ineffective, i.e. when $T = 0$. In such a case, the only allocation that is able to

ensure participation by both types of agents is the one that gives always all the good to type 1, i.e. $z = 1$. Consider the case $T > 0$. Define $\sigma(\beta, T) \equiv \frac{g_T(\beta)}{h_T(\beta)} = \frac{1-\beta^{T+2}}{(1+\beta)\left(1-\beta^{T+1}\right)}$. Notice that $\frac{f_T(\beta)}{h_T(\beta)} = \frac{\beta\left(1-\beta^T\right)}{(1+\beta)\left(1-\beta^{T+1}\right)} = 1-\sigma(\beta, T)$. Define also $\widehat{v}(\beta, T) \equiv \sigma(\beta, T) u(1) + (1-\sigma(\beta, T)) u(0)$ and $\widetilde{v}(\beta, T) \equiv (1-\sigma(\beta, T)) u(1) + \sigma(\beta, T) u(0)$. Finally, let

$$Z(\beta, T) \equiv \left\{ z \in [0,1] : z \geq u^{-1}(\widehat{v}(\beta, T)) \text{ and } z \leq 1 - u^{-1}(\widetilde{v}(\beta, T)) \right\},$$

be the set of allocations that satisfies (33) and (34) simultaneously. By Definition 3, this set identifies the allocations that can be sustained as a monetary equilibrium with discriminatory transfers.

Lemma 24 *$z(\mu)$ is a monetary equilibrium with discriminatory transfers if and only if $z(\mu) \in Z(\beta, T)$ for any $\beta \in (0,1)$ and $T > 0$.*

Proof. Consider a $z(\mu) \in Z(\beta, T)$. By definition, $Z(\beta, T)$ identifies, for any $\beta \in (0,1)$ and $T > 0$, the allocations on the frontier that satisfy (33) and (34), thus insuring agents' participation. By Lemma 23 any first best allocation can be achieved as a monetary equilibrium with discriminatory transfers, choosing transfers appropriately, ignoring agents' participation. Hence, $z(\mu) \in Z(\beta, T)$ is a monetary equilibrium with discriminatory transfers. Consider a $z(\mu) \notin Z(\beta, T)$. Some agent would have the incentive not to participate in the transfer scheme. Hence, such a $z(\mu)$ cannot be a monetary equilibrium with discriminatory transfers. ■

Let $Int(Z(\beta, T)) \equiv (u^{-1}(\widehat{v}(\beta, T)), 1 - u^{-1}(\widetilde{v}(\beta, T)))$ be the interior of $Z(\beta, T)$.

Proposition 8 *$Int(Z(\beta, T)) \neq \varnothing$ for any $\beta \in (0,1)$ and $T > 0$.*

Proof. $\sigma(\beta, T) u(1) + (1-\sigma(\beta, T)) u(0) < u(\sigma(\beta, T))$ and $(1-\sigma(\beta, T)) u(1) + \sigma(\beta, T) u(0) < u(1-\sigma(\beta, T))$ for any $\beta \in (0,1)$ and $T > 0$, by strict concavity of the utility function. Moreover, $u^{-1}(u(\sigma(\beta, T))) = \sigma(\beta, T)$, and $u^{-1}(u(1-\sigma(\beta, T))) = 1-\sigma(\beta, T)$. Therefore, $u^{-1}(\widehat{v}(\beta, T)) < \sigma(\beta, T) = 1-(1-\sigma(\beta, T)) < 1-u^{-1}(\widetilde{v}(\beta, T))$, for any $\beta \in (0,1)$ and $T > 0$. ■

Since the allocations $z(\mu)$ are on the first best frontier, when sustainable as monetary equilibria with discriminatory transfers, it is possible to Pareto improve upon the corresponding allocation without discriminatory transfers - which are not on the frontier, with the exception of the symmetric allocation.

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