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Poisson Search

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POISSON SEARCH*

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ABSTRACT. This paper presents a model of the job market in which the number of workers and companies is a Poisson random variable, as in Poisson games. The model has undominated equilibria that share some properties with directed search, while preserving some erratic elements typical of random search.

KEY WORDS. LABOR SEARCH, POISSON GAMES

1. INTRODUCTION

The process through which workers find jobs and companies fill vacancies is costly and time consuming. The search theory of the job market puts these costs and delays center stage.¹ Random search theory assumes that the matching of workers and companies is haphazard, with the worker and company locked in a bilateral monopoly negotiation, after being matched. The outcome is typically inefficient. At the other extreme, directed search theory assumes that the companies can post wage offers and the workers can direct their search towards the best offers. The outcome is typically constrained efficient.²

The random and directed versions of the theory capture different real world elements of the job search process, leading to distinct predictions for its outcome. For instance, the random version predicts that the distribution of the wages of

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¹ Diamond, Mortensen and Pissarides were the first to use search theory to model the difficulties of matching workers and companies. See, for instance, Diamond (1982); Pissarides (1985); Mortensen (1986), and the book *Equilibrium Unemployment Theory* by Pissarides (2000).

² The benchmark model is contained in Moen (1997), featuring a continuum of workers and companies. For a version with a finite number of workers and companies see Galenianos and Kircher (2012). Other directed search models are contained in Peters (2000) and Burdett et al (2001).

workers who change jobs should be independent of the previous wages,³ while the directed version predicts a systematic positive relationship. The available evidence shows a positive relationship with disturbances, pointing to a version of the job search theory with a blend of randomness and directedness.⁴ This paper offers one such version.

The key assumption of our model is that the number of workers and companies is a Poisson random variable, as in the Poisson games of Myerson (1998). This represents the random element of the model. On the other hand, as in the original directed search model of Moen (1997), there is a number of market places arranged by a market maker and distinguished by the wage that is being offered. The workers and companies can choose independently and simultaneously which market place to visit. This represents the directed element of the model. The job search activity is represented as a Poisson game with a finite but unknown number of possibly heterogeneous workers and companies. Our focus is on the undominated Nash equilibria of such a game.

We show that, at any undominated equilibrium, the workers and companies of higher types choose market places with higher wages. This is an immediate consequence of the main statistical property of Poisson games known as the environmental equivalence, whereby the posterior distribution of a player who has been selected to play the game over the number of other players in the game is the same as the prior distribution of an outsider over the number of all players. This property implies that the ranking of market places by the different types of workers and companies is driven immediately by the wage offered in each market place. Next, we prove that if there is only one market place, then, the equilibrium exists and is unique. In fact, when there are several market places, for each market place there exists an equilibrium in which that market place is the only one where companies and workers meet. On the other hand, an equilibrium with several market places is not always guaranteed to exist, and even when it exists may not be unique. When there exist equilibria with several market places, they coexist with the equilibria with each single market place.

This completes the analysis of the model with a market maker. Ultimately, however, we are interested in a more decentralized model of the job market. Thus, we examine a model in which, independently and simultaneously, the companies announce wage offers, the workers announce desired wages, and, if at least one company and one worker announce the same wage, then, a market emerges. In this context, we show that any equilibrium of the centralized model with the market places organized by the market maker can be replicated by an

³ See Godoy and Moen (2015).

⁴ As argued by Lentz and Moen (2017).

equilibrium of the decentralized model with wage announcements by the companies and workers, and, conversely, any equilibrium of the decentralized game can be replicated by an equilibrium of the centralized game. Hence, all the properties derived in the previous centralized model carry over to the decentralized one, similarly to Moen (1997).

The key idea of the paper is to conceive the job search activity as a situation in which the number of participants is a random variable, representing the uncertainty over the exact number of job applicants and companies with job openings. Since in non-cooperative games the number of players is common knowledge in the game, an extension of traditional game theory is needed to analyze situations in which the number of players is a random variable. Such games are in general difficult to handle, but the assumption that the number of players is a Poisson random variable makes them tractable, thanks to some convenient properties of the Poisson distribution.⁵

The Poisson game structure has been applied, so far, mostly to voting games, as shown by Myerson (2000, 2002). The idea is that in mass elections the voters ignore the exact size and identities of the electorate. In the macroeconomic literature, Poisson games have been studied in connection with speculative attacks on foreign currency markets when the number of speculators is unknown, by Makris (2008).⁶ We think of the job market as another situation in which it is sensible to assume that the exact overall number of participants is unknown. To the best of our knowledge, the structure of a Poisson game has been used in connection with search theory only in a paper by Norman (2016), for different purposes. The model has no heterogeneity among workers and companies and the Poisson game structure applies only to the worker side while the companies post wage offers to attract the workers, as in the directed search literature. The Poisson game is conceived as a device that helps endogenize the restriction to symmetric equilibria made in the directed search literature to justify the lack of coordination among workers that preserves the search frictions.⁷ In our paper, instead, the Poisson game structure is an alternative way of modelling the search friction. As a freebie, we get to analyze directly the equilibria of a finite game without symmetry restrictions. We do not attempt to review here the

⁵ Even the question what is the right solution concept for such games is far from obvious, see De Sinopoli et al (2014).

⁶ For applications of Poisson games to other economic environments see, for instance, Makris (2009); Ritzberger (2009); Jehiel and Lamy (2015).

⁷ With directed search, there are asymmetric equilibria in which each worker meets exactly one company, making the model frictionless, i.e. essentially Walrasian. The restriction to symmetric equilibria preserves the friction.

vast theoretical labor search literature, which is reviewed, for instance, in the Lectures in Labor Economics of Acemoglu and Autor (2020).

The paper proceeds as follows. Section 2 presents the model. Section 3 examines the matching framework. Section 4 derives the equilibria. Section 5 discusses the model with wage announcements. Section 6 concludes.

2. THE MODEL

We consider an economy in which *agents* are employers and workers, and we assume that their total number is a Poisson random variable with mean n . Thus, there are k agents with probability

$$P(k | n) = \frac{e^{-n} n^k}{k!}.$$

Agents can be of different *types*. There are I types of employers who differ in their productivity levels, and J types of workers who differ in their unemployment incomes. We denote the set of the $I+J$ types with $\mathcal{T}$ and a general element of $\mathcal{T}$ with t . With probability r_i a randomly sampled agent is an employer who has productivity y_i (i.e. he is of type i), where $i = 1, \dots, I$. While with probability r_j a randomly sampled agent is a worker who has unemployment income z_j (i.e. he is of type j), where $j = I+1, \dots, I+J$. We order employers' types according to productivity levels, letting $y_i < y_{i'}$ for $i < i'$, and workers' types according to unemployment incomes, letting $z_j < z_{j'}$ for $j < j'$. We denote the probability that an agent is an employer with $r_e = \sum_{i=1}^I r_i$, and the probability that an agent is a worker with $r_w = \sum_{j=I+1}^{I+J} r_j$, where $r_e + r_w = 1$. The decomposition property of the Poisson distribution implies that the number of agents of type t is a Poisson random variable with mean nr_t .

A market maker establishes a number K of *market places*, or simply *markets*, and sets for every market k the *wage* $w_k \in \mathbb{R}_+$. That is, all the jobs in a given market pay the same wage. We order markets according to wages, letting $w_k \leq w_{k'}$ for $k < k'$.

Each employer chooses if opening a vacancy in one of the market places, sustaining a sunk cost $c \geq 0$, or staying out from any market. We denote employers' action set with $A_i = A_e = \{1, \dots, K, O\}$ for every $i = 1, \dots, I$, where action k corresponds to choosing market k , while action O to staying out. On the other hand, each worker chooses if searching for a job in one of the market places or remaining unemployed. We denote workers' action set with $A_j = A_w = \{\bar{1}, \dots, \bar{K}, \bar{O}\}$ for every $j = I+1, \dots, I+J$, where $\bar{k}$ corresponds to choosing market k and $\bar{O}$ to not searching for a job. The *set of actions* is $A = A_e \cup A_w$. An *action profile* x is a vector that specifies for each action $a \in A$ the number of agents who choose that action, $x(a)$. The set of action profiles is $Z(A) \equiv \mathbb{Z}_+^{|A|}$. Thus, given a market k , $x(k)$

and $x(\bar{k})$ are respectively the number of employers and the number of workers who choose that market. We define $x_k = (x(k), x(\bar{k}))$ for every $k = 1, \dots, K$.

The number of matches in market k is determined by the *matching function* $f_k(x(k), x(\bar{k})) : \mathbb{Z}_+^2 \rightarrow \mathbb{Z}_+$. We assume $f_k(x(k), x(\bar{k})) = \min\{x(k), x(\bar{k})\}$. The reason of this assumption will be extensively discussed at the beginning of the next section. Then, we let

$$q_k(x(k), x(\bar{k})) = \frac{f_k(x(k), x(\bar{k}))}{x(k)}$$

denote the probability that an employer fills his vacancy entering market k , and

$$s_k(x(k), x(\bar{k})) = \frac{f_k(x(k), x(\bar{k}))}{x(\bar{k})}$$

denote the probability that a worker finds a job in k .

The *payoff* to an agent depends on his type, the action he chooses, and the number of other agents who choose each action. For every $x \in Z(A)$, $i = 1, \dots, I$, $j = I + 1, \dots, I + J$, $k = 1, \dots, K$ and $\bar{k} = \bar{1}, \dots, \bar{K}$, the payoff to an employer of type i is given by

$$\begin{aligned} u_i(O, x) &= 0, \\ u_i(k, x) &= q_k(x(k) + 1, x(\bar{k}))(y_i - w_k) - c \end{aligned}$$

while the payoff to a worker of type j is given by,

$$\begin{aligned} u_j(\bar{O}, x) &= z_j, \\ u_j(\bar{k}, x) &= s_k(x(k), x(\bar{k}) + 1)w_k + (1 - s_k(x(k), x(\bar{k}) + 1))z_j. \end{aligned}$$

Note that the payoff to an agent of entering a given market depends only on the number of other agents who also choose that market.

The set of *mixed actions* for agents of type t is given by $\Delta(A_t)$. As in Myerson (1998), a *strategy function* σ (or simply *strategy*) is an element of $\Sigma \equiv \{\sigma \in \Delta(A)^{I+J} : \sigma_t \in \Delta(A_t) \text{ for every } t \in \mathcal{T}\}$, mapping each type to the corresponding set of mixed actions. Strategy σ induces the *average behavior* $\tau(\sigma) \in \Delta(A)$ which is defined by $\tau(\sigma)(a) = \sum_{t \in \mathcal{T}} r_t \sigma_t(a)$ for each $a \in A$. When agents play according to σ , $\tau(\sigma)(a)$ is the probability that a randomly sampled player chooses action a . The number of agents who choose a is a Poisson random variable with mean $n\tau(\sigma)(a)$ and is independent of the number of agents that choose any other action. We let $\tau(\Sigma) \equiv \{\tilde{\tau} \in \Delta(A) : \tilde{\tau} = \tau(\sigma) \text{ for some } \sigma \in \Sigma\}$.⁸

⁸ It is clear that the same behavior $\tau \in \tau(\Sigma)$ can be induced by different strategy functions. However, τ is a sufficient statistics for the analysis of agents' optimal behavior, and we will often avoid to specify its dependence on σ . In fact, the payoff to an agent depends on the number of other agents who choose each action, independently of their specific types.

When the population's average behavior is summarized by $\tau \in \tau(\Sigma)$, the probability that in market k there are $x(k)$ employers and $x(\bar{k})$ workers is given by

$$P(x_k | \tau) = e^{-n(\tau(k) + \tau(\bar{k}))} \frac{[n\tau(k)]^{x(k)}}{x(k)!} \frac{[n\tau(\bar{k})]^{x(\bar{k})}}{x(\bar{k})!}.$$

Environmental equivalence implies that $P(x_k | \tau)$ is also the probability that each agent assigns to the event that there are $x(k)$ employers and $x(\bar{k})$ workers in market k . Then, the probability for an employer to fill his open vacancy if he enters market k is given by

$$Q_k(\tau(k), \tau(\bar{k})) = \sum_{x_k \in \mathbb{Z}_+^2} P(x_k | \tau) q_k(x(k) + 1, x(\bar{k})),$$

and his expected payoff if he is of type i is

$$U_i(k, \tau) = \sum_{x_k \in \mathbb{Z}_+^2} P(x_k | \tau) u_i(k, x) = Q_k(\tau(k), \tau(\bar{k}))(y_i - w_k) - c.$$

Similarly, the probability for a worker to find a job in market k is given by

$$S_k(\tau(k), \tau(\bar{k})) = \sum_{x_k \in \mathbb{Z}_+^2} P(x_k | \tau) s_k(x(k), x(\bar{k}) + 1),$$

and his expected payoff if he is of type j is

$$U_j(\bar{k}, \tau) = \sum_{x_k \in \mathbb{Z}_+^2} P(x_k | \tau) u_j(\bar{k}, x) = S_k(\tau(k), \tau(\bar{k}))w_k + [1 - S_k(\tau(k), \tau(\bar{k}))]z_j.$$

Definition 1. The strategy function σ is a *Nash equilibrium* if $U_t(\sigma_t | \tau(\sigma)) \geq U_t(\sigma'_t | \tau(\sigma))$ for every $t \in \mathcal{T}$ and $\sigma'_t \in \Delta(A_t)$.

We say that the average behavior $\tau \in \tau(\Sigma)$ is a *Nash equilibrium behavior* (or simply *Nash equilibrium*) if it is induced by a Nash equilibrium.

3. THE MATCHING FRAMEWORK

3.1. The matching function

We begin showing that the natural assumptions that the function $q_k(x(k), x(\bar{k}))$ is non-increasing in $x(k)$ and that the function $s_k(x(k), x(\bar{k}))$ is non-increasing in $x(\bar{k})$, which hold in Moen (1997), imply that the matching function must be either the min function or the min function with a threshold. Of course, we have $f_k(x(k), x(\bar{k})) \leq \min\{x(k), x(\bar{k})\}$ by definition, and we want $f_k(\cdot)$ not to be identically equal to zero.

Consider the first assumption. Since $x(k) \in \mathbb{Z}_+$, we have

$$q_k(x(k) + 1, x(\bar{k})) - q_k(x(k), x(\bar{k})) = \frac{f_k(x(k) + 1, x(\bar{k}))}{x(k) + 1} - \frac{f_k(x(k), x(\bar{k}))}{x(k)} \leq 0,$$

which is equivalent to

$$x(k)f_k(x(k)+1, x(\bar{k})) - x(k)f_k(x(k), x(\bar{k})) - f_k(x(k), x(\bar{k})) \leq 0. \quad (3.1)$$

If $f_k(x(k)+1, x(\bar{k})) > f_k(x(k), x(\bar{k})) + 1$ for some $x(k)$ and $x(\bar{k})$, in that case the lhs of (3.1) is strictly greater than

$$x(k)f_k(x(k), x(\bar{k})) + x(k) - x(k)f_k(x(k), x(\bar{k})) - f_k(x(k), x(\bar{k})) = x(k) - f_k(x(k), x(\bar{k})).$$

This last value is positive by definition of matching function, contradicting the assumption. So we have $f_k(x(k)+1, x(\bar{k})) \leq f_k(x(k), x(\bar{k})) + 1$ for every $x(k)$ and $x(\bar{k})$, that is, we have either $f_k(x(k)+1, x(\bar{k})) = f_k(x(k), x(\bar{k}))$ or $f_k(x(k)+1, x(\bar{k})) = f_k(x(k), x(\bar{k})) + 1$. Now, let $x(k) \leq x(\bar{k})$ and $f_k(x(k), x(\bar{k})) < x(k)$. If $f_k(x(k)+1, x(\bar{k})) = f_k(x(k), x(\bar{k})) + 1$ then the lhs of (3.1) is equal to

$$x(k) - f_k(x(k), x(\bar{k})) > 0,$$

which contradicts the assumption, so we have $f_k(x(k)+1, x(\bar{k})) = f_k(x(k), x(\bar{k}))$. An analogous reasoning based on the assumption that $s_k(x(k), x(\bar{k}))$ is non-increasing in $x(\bar{k})$ reveals that $f_k(x(k), x(\bar{k})+1) \leq f_k(x(k), x(\bar{k})) + 1$ for every $x(k)$ and $x(\bar{k})$ and that, if $x(\bar{k}) \leq x(k)$, $f_k(x(k), x(\bar{k})) < x(\bar{k})$ implies $f_k(x(k), x(\bar{k})+1) = f_k(x(k), x(\bar{k}))$.

Then, we have that $f_k(x(k), x(\bar{k})) \leq \min\{x(k), x(\bar{k})\}$ and that, for the above assumptions to be satisfied, whenever $f_k(x(k), x(\bar{k})) < \min\{x(k), x(\bar{k})\}$ it must be $f_k(x'(k), x(\bar{k})) = f_k(x(k), x'(\bar{k})) = f_k(x(k), x(\bar{k}))$ for all $x'(k) > x(k)$ and $x'(\bar{k}) > x(\bar{k})$. It is easy to see that $f_k(x(k), x(\bar{k})) = \min\{x(k), x(\bar{k})\}$ satisfies the two assumptions. Hence, we can conclude that the matching function can be either $f_k(x(k), x(\bar{k})) = \min\{x(k), x(\bar{k})\}$ or, for some threshold $\hat{x} \in \mathbb{Z}_+$, $f_k(x(k), x(\bar{k})) = \min\{x(k), x(\bar{k}), \hat{x}\}$.⁹ For simplicity, throughout the paper we will use the min function, but all the qualitative results remain valid for every threshold $\hat{x} \geq 1$.

Remark. Differently from Moen (1997), who considers a continuum of employers and workers, in our case the matching function is discrete. In that model the matching function is assumed to be homogeneous of degree one and concave. It is possible to see that discreteness implies that these two sole assumptions lead to the min function.¹⁰

⁹ Note that $q_k(x(k), x(\bar{k}))$ is not defined if $x(k) = 0$ while $s_k(x(k), x(\bar{k}))$ is not defined if $x(\bar{k}) = 0$. Of course, we have $f_k(0, 0) = f_k(n, 0) = f_k(0, n) = 0$ for every $n \in \mathbb{Z}_+$, and we have $f_k(1, n) = f_k(n, 1) = 1$ for every $n \geq 1$, otherwise $f_k(\cdot)$ is identically equal to zero.

¹⁰ In fact, under those assumptions, it cannot be $f_k(1, 1) = 0$ as it leads to the function identically equal to zero, since homogeneity would imply that $f_k(n, n) = 0$ for every n and then $f_k(n, n') > 0$ would contradict concavity. So it must be $f_k(1, 1) = 1$ and, by homogeneity, $f_k(n, n) = n$ for every n . But, then, $f_k(n, n') < \min\{n, n'\}$ would contradict concavity, so it must be $f_k(n, n') = \min\{n, n'\}$.

3.2. Matching probabilities

We now examine the behavior of the functions $Q_k(\cdot)$ and $S_k(\cdot)$ as their arguments vary.

First, we note that the probability that an employer fills his vacancy in market k when no other employer is expected to enter that market is given by

$$\begin{aligned} Q_k(0, \tau(\bar{k})) &= \sum_{x(k)=0}^{\infty} P(x(k) | 0) \sum_{x(\bar{k})=0}^{\infty} P(x(\bar{k}) | \tau(\bar{k})) q_k(x(k) + 1, x(\bar{k})) \\ &= \sum_{x(\bar{k})=0}^{\infty} P(x(\bar{k}) | \tau(\bar{k})) q_k(1, x(\bar{k})) = \sum_{x(\bar{k})=1}^{\infty} P(x(\bar{k}) | \tau(\bar{k})) = 1 - e^{-n\tau(\bar{k})}, \end{aligned}$$

which is the probability that at least one worker enters market k given $\tau(\bar{k}) \geq 0$. On the other hand, independently of other employers' behavior, if no worker is expected to enter market k then an employer will be matched in that market with probability zero, i.e.

$$\begin{aligned} Q_k(\tau(k), 0) &= \sum_{x(k)=0}^{\infty} P(x(k) | \tau(k)) \sum_{x(\bar{k})=0}^{\infty} P(x(\bar{k}) | 0) q_k(x(k) + 1, x(\bar{k})) \\ &= \sum_{x(k)=0}^{\infty} P(x(k) | \tau(k)) q_k(x(k) + 1, 0) = 0 \end{aligned}$$

for every $\tau(k) \geq 0$. Similar expressions define workers' probabilities of finding a job in market k . That is, when no other worker is expected to enter market k , a worker is matched with probability

$$S_k(\tau(k), 0) = 1 - e^{-n\tau(k)}$$

for every $\tau(k) \geq 0$, while when no employer opens a vacancy in market k then the worker is matched with probability zero, i.e.

$$S_k(0, \tau(\bar{k})) = 0$$

for every $\tau(\bar{k}) \geq 0$. We note also that the functions $Q_k(\cdot)$ and $S_k(\cdot)$ are continuous in both their arguments.

The following proposition shows that the probability for an employer to fill his vacancy in market k increases with the probability that workers enter k and decreases with the probability that other employers choose k . Similarly, the probability for a worker to find a job in market k increases with the probability that employers open their vacancies in k and decreases with the probability that other workers enter market k .

Lemma 1. *The function $Q_k(\tau(k), \tau(\bar{k}))$ is strictly decreasing in $\tau(k)$ and strictly increasing in $\tau(\bar{k})$, while the function $S_k(\tau(k), \tau(\bar{k}))$ is strictly increasing in $\tau(k)$ and strictly decreasing in $\tau(\bar{k})$.*

Proof. We can show that $Q_k(\tau(k), \tau(\bar{k}))$ is strictly decreasing in $\tau(k)$ by showing that, for every $\tau'(k) > \tau(k)$,

$$Q_k(\tau'(k), \tau(\bar{k})) - Q_k(\tau(k), \tau(\bar{k})) < 0. \quad (3.2)$$

In fact, the above inequality is equivalent to

$$\sum_{x(k) \in \mathbb{Z}_+} [P(x(k) | \tau'(k)) - P(x(k) | \tau(k))] \sum_{x(\bar{k}) \in \mathbb{Z}_+} P(x(\bar{k}) | \tau(\bar{k})) q_k(x(k) + 1, x(\bar{k})),$$

and the probability distribution over $\mathbb{Z}_+$ induced by $\tau'(k)$ first order stochastically dominates the one induced by $\tau(k)$, that is,

$$\sum_{x(k)=0}^{\bar{x}} P(x(k) | \tau'(k)) < \sum_{x(k)=0}^{\bar{x}} P(x(k) | \tau(k))$$

for every $\bar{x} \in \mathbb{Z}_+$.¹¹ Since $q_k(x(k) + 1, x(\bar{k}))$ is non-increasing in $x(k)$, inequality (3.2) follows. Likewise, $Q_k(\tau(k), \tau(\bar{k}))$ is strictly increasing in $\tau(\bar{k})$ because, for every $\tau'(\bar{k}) > \tau(\bar{k})$,

$$\begin{aligned} & Q_k(\tau(k), \tau'(\bar{k})) - Q_k(\tau(k), \tau(\bar{k})) = \\ &= \sum_{x(\bar{k}) \in \mathbb{Z}_+} [P(x(\bar{k}) | \tau'(\bar{k})) - P(x(\bar{k}) | \tau(\bar{k}))] \sum_{x(k) \in \mathbb{Z}_+} P(x(k) | \tau(k)) q_k(x(k) + 1, x(\bar{k})) > 0, \end{aligned}$$

given that the probability distribution over $\mathbb{Z}_+$ induced by $\tau'(\bar{k})$ first order stochastically dominates the one induced by $\tau(\bar{k})$ and the function $q_k(x(k) + 1, x(\bar{k}))$ is non-decreasing in $x(\bar{k})$.

In an analogous way it can be shown that the function $S_k(\tau(k), \tau(\bar{k}))$ is strictly increasing in the first argument and strictly decreasing in the second one, given that $s_k(x(k), x(\bar{k}))$ is non-decreasing in $x(k)$ and non-increasing in $x(\bar{k})$. $\square$

4. NASH EQUILIBRIA

In this section we analyze the undominated Nash equilibria of the model. We begin introducing the standard concepts of dominated action and dominated strategy.

Definition 2. Action σ_t is *dominated* by action σ'_t for agents of type t if $U_t(\sigma_t, \tau) \leq U_t(\sigma'_t, \tau)$ for every $\tau \in \tau(\Sigma)$ and $U_t(\sigma_t, \tau') < U_t(\sigma'_t, \tau')$ for some $\tau' \in \tau(\Sigma)$.

An action is dominated if there is another action that gives higher utility for every possible population average behavior and strictly higher utility for some. A strategy is dominated if it prescribes a dominated action for some type.

¹¹ In fact, $\sum_{k=0}^{\bar{k}} P(k | n') < \sum_{k=0}^{\bar{k}} P(k | n)$ for every $n' > n$ and $\bar{k} \in \mathbb{Z}_+$, and the decomposition property of Poisson games implies that the number of employers choosing k is a Poisson random variable.

Definition 3. The strategy function σ is *dominated* if there is a $t \in \mathcal{T}$ such that σ_t is a dominated action for players of type t .

We assume for simplicity that $w_1 > z_{I+J}$, that is, the lowest wage is larger than the highest unemployment income for workers. This assumption guarantees that, in every undominated strategy, all workers search for a job in some market. It is clear that the strategy function that prescribes every employer and every worker to stay out from any market is a Nash equilibrium. It follows from the above assumption that this autarkic equilibrium is dominated.

Lemma 2. *The average behavior τ such that $\tau(O) = r_e$ and $\tau(\bar{O}) = r_w$ is a dominated Nash equilibrium.*

We say that market k is *active* given the average behavior τ if at least some employers and some workers enter k with positive probability, that is, $\tau(k) > 0$ and $\tau(\bar{k}) > 0$. Recall that environmental equivalence implies that, for every average behavior τ and market k , the probability to fill the vacancy in k is the same for every employer, and the probability to find a job in k is the same for every worker. Consider an equilibrium behavior τ such that markets k and k' are both active, with $w_k < w_{k'}$. Since some employers choose market k' in equilibrium, the higher wage they have to pay must be compensated by a larger probability of being matched in that market. Analogously, since some workers enter market k in equilibrium, the probability to find a job there must be higher.¹² We can therefore state the following.

Lemma 3. *Let markets k and k' be active given the Nash equilibrium τ , and let $w_k < w_{k'}$. Then $Q_k(\tau(k), \tau(\bar{k})) < Q_{k'}(\tau(k'), \tau(\bar{k}'))$ and $S_k(\tau(k), \tau(\bar{k})) > S_{k'}(\tau(k'), \tau(\bar{k}'))$.*

Then, we can show that in every undominated equilibrium employers and workers of higher types choose markets that pay higher wages, consistently with Moen (1997). First we prove that, given an equilibrium behavior of the population and two active markets, if an employer prefers to enter the market that pays the higher wage then all the employers with a larger productivity level will also do so (and the larger the productivity, the larger the gain of choosing that market rather than the other). Likewise, if an employer prefers to enter the market that pays the lower wage, then the same is true for all the employers with smaller productivity levels (and the smaller the productivity, the larger the gain of choosing that market rather than the other).¹³ Recall that $y_i < y_{i'}$ for $i < i'$.

¹² Clearly, if two markets k and k' pay the same wages and are both active in equilibrium then, for every agent, the probability to be matched must be the same in k and in k' .

¹³ Of course, given any population behavior, if an employer prefers to enter some market rather than staying out then all the employers of higher types will also do so, while if he prefers

Lemma 4. *Let τ be a Nash equilibrium, let k and k' be two active markets given τ with $w_k < w_{k'}$, and let $i < i'$. If $U_i(k', \tau) \geq U_i(k, \tau)$ then $U_{i'}(k', \tau) > U_{i'}(k, \tau)$, while if $U_{i'}(k, \tau) \geq U_{i'}(k', \tau)$ then $U_i(k, \tau) > U_i(k', \tau)$.*

Proof. We can prove that

$$U_{i'}(k', \tau) - U_{i'}(k, \tau) > U_i(k', \tau) - U_i(k, \tau),$$

which implies both results. Let us write for simplicity Q_k and $Q_{k'}$ instead of $Q_k(\tau(k), \tau(\bar{k}))$ and $Q_{k'}(\tau(k'), \tau(\bar{k}'))$. The above inequality is equivalent to

$$Q_{k'}(y_{i'} - w_{k'}) - Q_k(y_{i'} - w_k) > Q_{k'}(y_i - w_{k'}) - Q_k(y_i - w_k).$$

Rearranging terms, we have

$$Q_{k'} > Q_k,$$

which is satisfied by Lemma 3. $\square$

Analogously, given an equilibrium behavior and two active markets, if a worker prefers to enter the market that pays the larger wage then all the workers of higher types will do so (and the higher the type the higher the gain of choosing that market rather than the other). On the other hand, if a worker prefers to enter the market that pays the lower wage then the same holds for every worker with lower unemployment income (and the lower the type, the lower the gain of choosing that market rather than the other). Recall that $z_j < z_{j'}$ for every $j < j'$.

Lemma 5. *Let τ be a Nash equilibrium, let k and k' be two active markets given τ with $w_k < w_{k'}$, and let $j < j'$. If $U_j(\bar{k}', \tau) \geq U_j(\bar{k}, \tau)$ then $U_{j'}(\bar{k}', \tau) > U_{j'}(\bar{k}, \tau)$, while if $U_{j'}(\bar{k}, \tau) \geq U_{j'}(\bar{k}', \tau)$ then $U_j(\bar{k}, \tau) > U_j(\bar{k}', \tau)$.*

Proof. As before, we can show that

$$U_{j'}(\bar{k}', \tau) - U_{j'}(\bar{k}, \tau) > U_j(\bar{k}', \tau) - U_j(\bar{k}, \tau),$$

which implies both results. Writing S_k and $S_{k'}$ instead of $S_k(\tau(k), \tau(\bar{k}))$ and $S_{k'}(\tau(k'), \tau(\bar{k}'))$, the above inequality is equivalent to

$$S_{k'}w_{k'} + (1 - S_{k'})z_{j'} - S_kw_k - (1 - S_k)z_{j'} > S_{k'}w_{k'} + (1 - S_{k'})z_j - S_kw_k - (1 - S_k)z_j,$$

and rearranging terms we have

$$S_{k'} < S_k,$$

which holds by Lemma 3. $\square$

to stay out rather than entering any market then the same is true for all the employers of lower types.

Given that, in equilibrium, employers' and workers' types are ordered and agents will not choose to enter markets in which the probability of being matched is zero, it seems convenient to perform the equilibrium analysis fixing the number of active markets. We start examining equilibria such that only one market place is active. Then, we consider equilibria in which matching between employers and workers occurs with positive probability in a couple of market places. This will provide us information on the characteristics of the equilibria for a general number of active markets and reveal some useful economic insights.

4.1. One active market

Consider the case in which only market k is active in equilibrium. This implies that no agent chooses any other market with positive probability, since the probability of being matched there is zero. Specifically, every undominated strategy prescribes all the workers to choose market k with probability 1. On the other hand, at least some employers will enter market k with positive probability in equilibrium. In particular, by Lemma 4, there exists a type i^* that separates employers entering the market and employers staying out, so two possible cases can occur. There can be a type i^* who is indifferent between entering market k or staying out, and all types $i > i^*$ enter k while all types $i < i^*$ stay out. Or there is a type i^* such that all types $i \geq i^*$ enter market k while all types $i < i^*$ stay out (it can be $i^* = 1$, i.e. all employers choose market k with probability 1). Thus, characterizing the equilibria in this case coincides with characterizing the type i^* .

To this end, take the type of employers with the lowest productivity level y_1 , and consider the average behavior τ such that all workers enter market k , i.e. $\tau(\bar{k}) = r_w$, as well as all and only the employers with productivity larger than y_1 , i.e. $\tau(k) = r_e - r_1$. The expected payoff to that type of entering market k is equal to

$$Q_k(r_e - r_1, r_w)(y_1 - w_k) - c. \quad (4.1)$$

If this is positive, then type 1 employers will choose to enter the market with some positive probability in equilibrium. In the worst case in which every other employer enters the market, so that $Q_k(\cdot, r_w)$ reaches its minimum value by Lemma 1, their expected payoff is

$$Q_k(r_e, r_w)(y_1 - w_k) - c.$$

If this value is positive, then they will choose to enter market k with probability 1. If it is negative, then they will choose to enter k with probability $\alpha \in [0, 1]$, where α solves

$$Q_k(r_e - (1 - \alpha)r_1, r_w)(y_1 - w_k) - c = 0.$$

Since the function $Q_k(\cdot, r_w)$ is continuous and strictly decreasing, α exists and is unique. In both cases, all the employers with higher productivity levels will enter market k with probability 1. Thus, in the first case there is an equilibrium behavior τ^* such that $\tau^*(k) = r_e$ and $\tau^*(\bar{k}) = r_w$, while in the second case we have $\tau^*(k) = r_e - (1 - \alpha)r_1$, $\tau^*(O) = (1 - \alpha)r_1$, and $\tau^*(\bar{k}) = r_w$.

On the other hand, if the value of (4.1) is strictly negative then all the employers of type 1 will choose to stay out. Then, the incentives of employers with productivity y_2 must be examined. In particular, if $Q_k(r_e - r_1, r_w)(y_2 - w_k) - c \geq 0$ then they will choose to enter market k with probability 1 in equilibrium, as well as all the employers of higher types. Hence, in this case there is an equilibrium behavior τ^* such that $\tau^*(k) = r_e - r_1$, $\tau^*(O) = r_1$ and $\tau^*(\bar{k}) = r_w$. If $Q_k(r_e - r_1, r_w)(y_2 - w_k) - c \leq 0$ and $Q_k(r_e - r_1 - r_2, r_w)(y_2 - w_k) - c \geq 0$ then type 2 employers will choose to enter market k with probability $\beta \in [0, 1]$, where β is computed in a similar way as the above α , while all employers of higher types will enter with probability 1. Therefore, we have $\tau^*(k) = r_e - r_1 - (1 - \beta)r_2$, $\tau^*(O) = r_1 + (1 - \beta)r_2$, and $\tau^*(\bar{k}) = r_w$. If $Q_k(r_e - r_1 - r_2, r_w)(y_2 - w_k) - c < 0$ then type 2 employers will choose to stay out, and incentives of employers with productivity y_3 must be examined, and so on.

This analysis holds for any given market place. It implies that an equilibrium with only one active market exists whenever firms are productive enough.¹⁴ In fact, it exists for each market k such that $Q_k(0, r_w)(y_I - w_k) - c > 0$, that is, such that at least the highest productivity employers are willing to enter k with some strictly positive probability when all the workers enter the market. We assume that this is true for every market, so that each market place in the economy has a *raison d'être*.

The next proposition establishes that the undominated equilibrium in which a given market place is the only active one is unique.

Proposition 1. *For each market k , there exists a unique Nash equilibrium such that only market k is active.*

Proof. Suppose to the contrary that there are two equilibria such that only market k is active, inducing behaviors τ_1^* and τ_2^* . We have $\tau_1^*(\bar{k}) = \tau_2^*(\bar{k}) = r_w$. Let x be the probability that a randomly sampled agent is an employer who enters market k in the first equilibrium, i.e. $x = \tau_1^*(k)$. Then, by Lemma 4, there is a type i_1^* such that either

$$\begin{aligned} Q_k(x, r_w)(y_i - w_k) - c &< 0 & \text{for } i < i_1^*, \\ Q_k(x, r_w)(y_i - w_k) - c &> 0 & \text{for } i \geq i_1^*, \end{aligned}$$

¹⁴ The same is true in Moen (1997).

where $x = \sum_{i=i_1^*}^I r_i$, or

$$Q_k(x, r_w)(y_{i_1^*} - w_k) - c = 0,$$

where $x = \alpha r_{i_1^*} + \sum_{i=i_1^*+1}^I r_i$ for some $\alpha \in [0, 1]$. Likewise, let y be the probability that an agent is an employer who enters market k in the second equilibrium, i.e. $y = \tau_2^*(k)$, and let i_2^* be the type such that either

$$Q_k(y, r_w)(y_i - w_k) - c < 0 \quad \text{for } i < i_2^*,$$

$$Q_k(y, r_w)(y_i - w_k) - c > 0 \quad \text{for } i \geq i_2^*,$$

where $y = \sum_{i=i_2^*}^I r_i$, or

$$Q_k(y, r_w)(y_{i_2^*} - w_k) - c = 0,$$

where $y = \beta r_{i_2^*} + \sum_{i=i_2^*+1}^I r_i$ for some $\beta \in [0, 1]$.

Let $y > x$ (a similar argument applies to the case $y < x$). By Lemma 1, we have $Q_k(y, r_w) < Q_k(x, r_w)$. Hence, for the above conditions of the two equilibria to simultaneously hold, it must be $i_1^* < i_2^*$. But this implies $y < x$, which leads to a contradiction. Therefore, the desired result follows. $\square$

Thus, given an economy with K market places, there exist K equilibria, all undominated, in which each market place is the only active one. Moreover, these equilibria respect all the basic strategic principles one can hope for, being stable sets as singletons.¹⁵

4.2. Two active markets

We study now the case in which two market places are active in equilibrium, which will provide us also more general intuitions.

Let the two active markets be k and k' , with $w_k < w_{k'}$. In any undominated equilibrium all the workers will be distributed over them, while employers may also choose to stay out. Thus, given an average behavior τ , let s be the probability that a randomly selected agent is a worker who enters market k , and $r_w - s$ be the probability he is a worker who enters market k' . That is, $s = \tau(\bar{k})$ and $r_w - s = \tau(\bar{k}')$. Moreover, let x be the probability that an agent is an employer who enters market k , y be the probability he is an employer who enters market k' , and $r_e - x - y$ be the probability he is an employer staying out of any market.

¹⁵ It is not difficult to see that each of these equilibria is a stable set as defined in De Sinopoli et al (2014). In broad terms, a stable set in Poisson games is a minimal subset of Nash equilibria such that every close-by game obtained through perturbations of the average behavior has a Nash equilibrium close to the stable set. The equilibria under consideration are such that best responses are strict for all types except at most one (that is, the type of employers who is indifferent between entering the market and staying out). If this type's best response is strict then the result follows readily, otherwise note that any perturbation of the average behavior in the definition of stable set can be compensated by his mixed action.

That is, $x = \tau(k)$, $y = \tau(k')$, and $r_e - x - y = \tau(O)$. We begin discussing the incentives of workers and those of employers, relying on the results stated in Lemmas 4 and 5, and then we explore the equilibrium they lead to.

4.2.1. *Workers.* Given the behavior of employers, summarized by x and y , we need to determine the type j^* that divides workers choosing to enter market k and workers choosing to enter market k' .

To this end, we can start considering the type of workers with the largest unemployment income z_J . Since markets k and k' are both active in equilibrium, either workers of this type are indifferent between the two markets or they strictly prefer to enter k' rather than k . In particular, they will be indifferent if their expected payoff of choosing market k is larger than that of choosing market k' when all and only the workers of the highest type enter k' , and their expected payoff of choosing market k is strictly smaller than that of choosing market k' when no other worker is there (which is the necessary condition for market k' to be active). That is,

$$S_k(x, r_w - r_J)(w_k - z_J) \geq S_{k'}(y, r_J)(w_{k'} - z_J)$$

and

$$S_k(x, r_w)(w_k - z_J) < S_{k'}(y, 0)(w_{k'} - z_J).$$

In this case, they will enter market k' with probability $\alpha \in (0, 1]$ such that

$$S_k(x, r_w - \alpha r_J)(w_k - z_J) = S_{k'}(y, \alpha r_J)(w_{k'} - z_J),$$

and therefore we have $s = r_w - \alpha r_J$. Note that, for any given x and y , α exists and is unique by Lemma 1.

On the other hand, workers of type J enter market k' with probability 1 if

$$S_k(x, r_w - r_J)(w_k - z_J) < S_{k'}(y, r_J)(w_{k'} - z_J).$$

Then, to determine the type j^* , a similar analysis needs to be done for workers of smaller types.

4.2.2. *Employers.* Given workers' behavior summarized by s , we need now to determine two types of employers. The type i^* that separates employers staying out from any market and employers entering one of them, and the type l^* that separates employers entering market k' and employers who either enter market k or stay out.¹⁶

¹⁶ Note that i^* and l^* may coincide, either when there is a type who is indifferent between staying out and choosing k (and possibly also choosing k'), all lower types staying out while all higher types choosing k' , or when there is a type who is indifferent between the two markets, all lower types staying out while all higher types choosing k' .

Let y be fixed. If $y > r_e - r_1$, then in equilibrium employers with the lowest productivity y_1 must be indifferent between the two markets, while all employers of higher types choose to enter market k' , so we have $i^* = l^* = 1$. In particular, it will be either

$$Q_k(r_e - y, s)(y_1 - w_k) = Q_{k'}(y, r_w - s)(y_1 - w_{k'}) \geq c$$

so that no employer chooses to stay out and $x = r_e - y$, or

$$Q_k(\alpha r_1, s)(y_1 - w_k) = Q_{k'}(y, r_w - s)(y_1 - w_{k'}) = c$$

for some $\alpha \in [0, 1]$ with $\alpha r_1 < r_e - y$, so that type 1 employers randomize with positive probabilities over staying out, entering market k and entering market k' , with $x = \alpha r_1$.

Then, let $y \leq r_e - r_1$, consider employers of type 1 and take their expected payoff of entering market k when no employer stays out,

$$Q_k(r_e - y, s)(y_1 - w_k) - c.$$

If this is positive, then in equilibrium they choose to enter market k with probability 1 and no employer stays out, so we have $i^* = 1$ and $x = r_e - y$. If that payoff is negative, but their expected payoff of entering market k when all and only type 1 employers stay out is positive, i.e.

$$Q_k(r_e - r_1 - y, s)(y_1 - w_k) - c \geq 0,$$

then they are indifferent between staying out and entering market k . In this case they choose to stay out with probability $\beta \in [0, 1]$ such that

$$Q_k(r_e - \beta r_1 - y, s)(y_1 - w_k) - c = 0,$$

where β exists and is unique by Lemma 1. So we have $i^* = 1$ and $x = r_e - \beta r_1 - y$. On the other hand, if $Q_k(r_e - r_1 - y, s)(y_1 - w_k) - c < 0$ then type 1 employers will choose to stay out with probability 1, and employers of higher types must be considered to determine i^* .

Now, let x be fixed and consider the employers with the largest productivity level y_I . Note that, for market k' to be active, it must be

$$Q_k(x, s)(y_I - w_k) < Q_{k'}(0, r_w - s)(y_I - w_{k'}).$$

If $x > r_e - r_I$ then, in equilibrium, those employers must be indifferent between entering market k or market k' , and all employers of lower types choose either to enter market k or to stay out, so we have $l^* = I$. Precisely, employers of type I choose market k' with probability $\gamma \in (0, 1)$ such that

$$Q_k(x, s)(y_I - w_k) = Q_{k'}(\gamma r_I, r_w - s)(y_I - w_{k'}), \quad (4.2)$$

so we have $y = \gamma r_I$.

Then, let $x \leq r_e - r_I$. If type I employers get a higher payoff by choosing market k rather than market k' when all and only the employers of their same type enter k' , that is, if

$$Q_k(x, s)(y_I - w_k) \geq Q_{k'}(r_I, r_w - s)(y_I - w_{k'}),$$

then, as before, in equilibrium they are indifferent between the two markets. So we have $l^* = I$ and $y = \gamma r_I$, where $\gamma \in (0, 1]$ solves (4.2) and exists and is unique by Lemma 1. On the other hand, if

$$Q_k(x, s)(y_I - w_k) < Q_{k'}(r_I, r_w - s)(y_I - w_{k'})$$

then, in equilibrium, type I employers will enter market k' with probability 1 and incentives of employers of lower types need to be analyzed to determine l^* .

4.2.3. Equilibrium. The above analysis leads to the value of s that, for any given x and y , satisfies workers' equilibrium condition, which characterizes the type j^* that separates workers entering market k and workers entering market k' . Such a condition can be of the form

$$S_k(x, s)(w_k - z_{j^*}) = S_{k'}(y, r_w - s)(w_{k'} - z_{j^*}) \quad (4.3a)$$

or of the form

$$\begin{aligned} S_k(x, s)(w_k - z_j) &> S_{k'}(y, r_w - s)(w_{k'} - z_j) & \text{for } j < j^* \\ S_k(x, s)(w_k - z_j) &< S_{k'}(y, r_w - s)(w_{k'} - z_j) & \text{for } j \geq j^*, \end{aligned} \quad (4.3b)$$

where $s = \sum_{j=1}^{j^*} r_i - \alpha r_{j^*}$ for $\alpha \in [0, 1]$.

Furthermore, for any given s , there are values of x and y that satisfy employers' equilibrium conditions. Precisely, we have the condition that characterizes the type i^* who separates employers staying out and entering market k , which can be either of the form

$$Q_k(x, s)(y_{i^*} - w_k) - c = 0 \quad (4.4a)$$

or of the form

$$\begin{aligned} Q_k(x, s)(y_i - w_k) - c &< 0 & \text{for } i < i^* \\ Q_k(x, s)(y_i - w_k) - c &> 0 & \text{for } i \geq i^*; \end{aligned} \quad (4.4b)$$

and the condition that characterizes the type l^* who separates employers choosing market k and choosing market k' , which can be either of the form

$$Q_k(x, s)(y_{l^*} - w_k) = Q_{k'}(y, r_w - s)(y_{l^*} - w_{k'}) \quad (4.5a)$$

or of the form

$$\begin{aligned} Q_k(x, s)(y_i - w_k) &> Q_{k'}(y, r_w - s)(y_i - w_{k'}) & \text{for } i < l^* \\ Q_k(x, s)(y_i - w_k) &< Q_{k'}(y, r_w - s)(y_i - w_{k'}) & \text{for } i \geq l^*, \end{aligned} \quad (4.5b)$$

where

$$x = \sum_{i=i^*}^{l^*} r_i - \beta r_{i^*} - \gamma r_{l^*} \quad \text{and} \quad y = \gamma r_{l^*} + \sum_{i=l^*+1}^I r_i$$

for $\beta \in [0, 1]$ and $\gamma \in [0, 1]$.¹⁷

Either condition (4.3a) or (4.3b) determines s as a continuous function of x and y . By Lemma 1, $s(x, y)$ is weakly increasing in x and weakly decreasing in y . As economic intuition suggests, the higher the probability employers choose one of the two markets, the higher the probability workers will also choose that market in equilibrium. Weakness derives from the fact that, when the “separating” type is not indifferent between the two markets, a small change in employers’ behavior does not imply a change in workers’ one. On the other hand, if the separating type is indifferent between the two markets, then s must adjust whenever x and y vary in order to maintain the equilibrium equality. Moreover, for any given s , either condition (4.4a) or (4.4b) determines the value of $x(s)$, which is a continuous and weakly increasing function.¹⁸ That is, if the probability workers enter market k increases then the probability employers choose to enter that market rather than staying out in also increases. Then, y is determined by either condition (4.5a) or (4.5b). The function $y(s)$ is continuous and weakly decreasing, i.e. the higher the probability workers enter market k , the lower the probability employers will choose to enter market k' .

An equilibrium is given by values s^* , x^* and y^* for which the conditions above are simultaneously satisfied. Hence, it is the solution to a system of 3 equations (or 2 inequalities in place of each of them) and 3 unknowns. However, such a system does not always have a solution, so an equilibrium does not always exist. We now illustrate an example in which there is no equilibrium where two given markets are active even if the two equilibria where just one of the markets is active exist.

Example 1. Let us consider the simplest case in which there is only one type of employers, i , and one type of workers, j , and take two markets k and k' . We assume that the probability of being an employer is close to zero and the probability of being a worker is close to 1. This implies that the probability for

¹⁷ Note that if $w_k = w'_k$ then we have $S_k(x, s) = S_{k'}(y, r_w - s)$ and $Q_k(x, s) = Q_{k'}(y, r_w - s)$. In particular, all types of workers will be indifferent between the two markets, while employers’ types will be divided into those indifferent between the two markets (who may strictly prefer entering one of them rather than staying out or may be indifferent between the three choices) and those who strictly prefer to stay out.

¹⁸ As s increases, if the separating type is indifferent between entering k and staying out then x increases so that the probability $Q_k(x, s)$ remains constant, while if the separating type strictly prefers entering market k then x does not change so $Q_k(x, s)$ increases.

employers of being matched in a market when all workers are there is the highest and overcomes alone the cost of opening the vacancy, thus guaranteeing the existence of the equilibria in which only one of the markets is active. Moreover, we assume that the wages paid in the two markets are similar and we let the difference between employers' productivity and these wages be slightly larger than the cost. So, when workers distribute over the two markets, in the one they choose with lower probability, the probability of matching for employers multiplied by that difference does no more overcome the cost.

Thus, let $y_i = 3$, $z_j = 1$, $c = 0.9$, and let the expected number of agents be $n = 4$. To avoid useless computations, let $w_k = w_{k'} = 2$, $r_e = 0$, and $r_w = 1$. Since the functions Q_k and S_k are continuous in their arguments and so are the equilibrium conditions, which will involve strict inequalities, the results of this "degenerate" case will remain valid in a neighborhood of such parameters values. Consider the equilibrium in which just one of the two markets, say k , is active. Workers choose to enter k with probability 1, and so do employers as long as

$$Q_k(r_e, r_w)(y_i - w_k) \geq c,$$

which is satisfied because $Q_k(0, 1) = 1 - e^{-4} \approx 0.981 > 0.9$. Likewise, there exists the equilibrium in which workers and employers enter market k' with probability 1. Now, suppose both markets k and k' are active in equilibrium. Workers will distribute over them, entering k with probability $\sigma_j(\bar{k})$ and k' with probability $\sigma_j(\bar{k}') = 1 - \sigma_j(\bar{k})$. Let $\sigma_j(\bar{k}) \leq \sigma_j(\bar{k}')$, that is, $\sigma_j(\bar{k}) \leq \frac{1}{2}$. If $r_w = 1$, we have $\tau(\bar{k}) = \sigma_j(\bar{k})$ and

$$Q_k(0, \tau(\bar{k})) = 1 - e^{-4\tau(\bar{k})} < 1 - e^{-2} \approx 0.865 < 0.9.$$

Since the expected payoff to employers of entering market k is strictly negative, they now prefer to stay out. It follows that an equilibrium in which both markets are active does not exist.

In this example, an equilibrium where the two markets are both active does not exist because the cost for employers of opening a vacancy is too high. In fact we can prove that, in the same simplest case with one type of employers and one type of workers, for any couple of markets an equilibrium in which they are both active exists whenever that cost is sufficiently small relative to other fundamentals.

Proposition 2. *Let $I = J = 1$. For each couple of markets k and k' , there exists a Nash equilibrium such that only markets k and k' are active whenever c is sufficiently small.*

Proof. Let i be the unique type of employers and j the unique type of workers, consider markets k and k' , and suppose that $c = 0$. Consider an equilibrium such

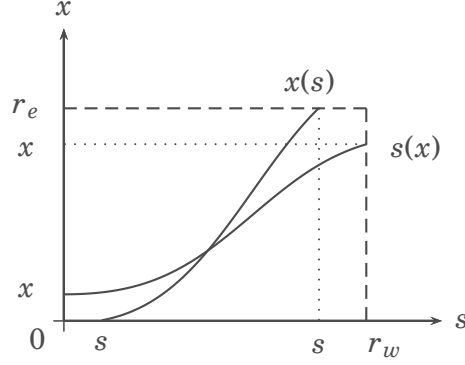


FIGURE 1. Existence of an equilibrium with two active markets when $I = J = 1$.

that k and k' are the only active markets. It is clear that employers and workers must be indifferent between entering market k or market k' , so we have two equilibrium conditions,

$$\frac{Q_k(x, s)}{Q_{k'}(r_e - x, r_w - s)} = \frac{y_i - w_{k'}}{y_i - w_k}$$

and

$$\frac{S_k(x, s)}{S_{k'}(r_e - x, r_w - s)} = \frac{w_{k'} - z_j}{w_k - z_j},$$

where x and s are the probabilities that a random agent is respectively an employer and a worker entering market k . The first condition determines x as a continuous and strictly increasing function of s , while the second condition determines s as a continuous and strictly increasing function of x .

Since $\frac{y_i - w_{k'}}{y_i - w_k}$ is strictly positive, when $s = 0$ or $s = r_w$ there is no value of x for which the first condition is satisfied.¹⁹ Similarly, since $\frac{w_{k'} - z_j}{w_k - z_j} > 0$, when $x = 0$ or $x = r_e$ there is no value of s for which the second condition is satisfied. On the other hand, recall that $Q_k(0, s) > 0$ for every $s > 0$ and $S_k(x, 0) > 0$ for every $x > 0$. Hence, $x(s)$ is defined over $s \in [s, s]$, with $0 < s < s < r_w$, and ranges from 0 to r_e . While $s(x)$ is defined over $x \in [x, x]$, with $0 < x < x < r_e$, and ranges from 0 to r_w . Figure 1 depicts in the x - s plane two functions $x(s)$ and $s(x)$ with the characteristics above and reveals that, regardless of their precise shapes, they must intersect at least once.

This implies that at least an equilibrium exists whenever $c = 0$. Note that, since in equilibrium the probability for employers of being matched in the two markets is strictly positive, we have $Q_k(x, s)(y_i - w_k) > 0$. That is, employers prefer strictly to enter one of the two markets rather than staying out. Thus, the

¹⁹ In fact we have, respectively, $Q_k(x, s) = 0$ and $Q_{k'}(r_e - x, r_w - s) = 0$.

existence result remains valid for values of c in a sufficiently small neighborhood of 0. $\square$

Given an economy with K market places, there exist at least K equilibria. In fact, for each market, there exists a unique equilibrium in which only that market is active. Existence of equilibria where more than one market place is active may demand more stringent requirements. Whenever these equilibria exist, they are such that employers and workers of higher types choose to enter the markets that pay larger wages.

5. WAGE ANNOUNCEMENTS

In the same vein as Moen (1997), we now show that the equilibria of the model in which wages are established by the market maker can be realized if it is assumed that wages are announced by the agents.

Let us consider again a Poisson game with expected number of players n , I types of employers and J types of workers whose characteristics are as defined in Section 2. But we now assume that agents directly announce the desired wage, or decide not to announce any wage. Thus, their action sets are $A_e = A_w = \mathbb{R}_+ + O$, where action O corresponds to not announcing any wage. A market is formed when at least one employer and one worker post the same wage. As in the original model, the number of matches in a market is given by the minimum between the number of employers and the number of workers who choose the corresponding wage. The payoffs to agents are as defined in Section 2, taking appropriately into account the change in their action sets.

Consider an equilibrium of the original model in which the set of active markets is $\tilde{K}$. Recall that in such an equilibrium no agent chooses with positive probability a market that is not active. In the model with wage announcements, the equivalent strategy combination is such that each type entering market $k \in \tilde{K}$ (resp. staying out from any market) with strictly positive probability posts the corresponding wage w_k (resp. does not announce any wage) with that same probability. Then, an agent who expects opponents to play according to this combination has no incentive to deviate and choose a different wage $w_{k'}$ with $k' \in \tilde{K}$, because the original profile is an equilibrium. Moreover, no agent has incentive to choose any other wage either, that is, a wage that no market $k \in \tilde{K}$ pays, since the probability of being matched in that case would be zero. It follows that every equilibrium of the original model is an equilibrium also in the model with wage announcements.

Take now an equilibrium of the model with wage announcements with finite support $\tilde{W} \subset \mathbb{R}_+$, and consider the original model in which the market maker establishes the corresponding set of markets; that is, for every wage $w \in \tilde{W}$ there is

a market that pays w . The equivalent strategy combination in this model is such that each type posting salary w (resp. not posting any salary) with strictly positive probability chooses to enter the market that pays w (resp. chooses to stay out) with that same probability. It is clear that, under such a strategy combination, no agent has incentive to deviate to any other action because the profile under consideration is an equilibrium. On the other hand, every equilibrium of the model with wage announcements with infinite support is equivalent to the autarkic equilibrium of the original model, since no matching occurs.²⁰ We can conclude that every equilibrium of the model with wage announcements is also induced by the original model.

Remark. Moen (1997) analyzes the case of wage announcements assuming a two-stage process, in which first employers announce the wage they are going to pay and then workers apply for a job that pays a given wage. The Poisson setting forces us to consider simultaneous announcements by employers and workers. In fact, Poisson games have been defined by Myerson (1998) as simultaneous games, in which players do not observe the realization of their opponents but have only some probabilistic information about their number and characteristics. The only conceivable sequential process would involve, first, a Poisson game in which employers announce wages and, then, another Poisson game in which workers choose one of those wages knowing employers' realization and choices.²¹ However, such a model could not be compared with our original one, in which agents' strategies do not depend on the precise realization of opponents' behavior but only on their expectation of this behavior. Note that in Moen's model with a continuum of agents this distinction is not relevant, as expectations and realizations coincide when the number of agents is infinite. In the Poisson framework, the equivalence, in terms of inducing the same equilibria, of the original model and the model with wage announcements requires them to be either both simultaneous games or both two-stage processes.

²⁰ Note that if some employers and some workers randomize with atomless probability distributions over a continuum of wages, the probability for an agent of being matched if he chooses a wage in that continuum is zero. Since employers sustain a positive cost of posting, they strictly prefer not announcing any wage rather than announcing a wage in the continuum, while workers are indifferent between the two choices. Thus, any equilibrium with infinite support is such that (possibly some) workers randomize with atomless probability distribution over a continuum of wages, while employers do not post any wage, so every agent is matched with probability zero.

²¹ A similar model is considered in Norman (2016).

6. CONCLUSION

This paper has presented a new model of the job market based on the Poisson representation of a game with population uncertainty. The game is finite and the Poisson structure allows to analyze the equilibria in a fairly simple way. The undominated equilibria have properties that place the model in between the random and directed search model typically used in the labor literature.

REFERENCES

- Acemoglu D. and D. Autor. Lectures in labor economics. *Mimeo*, 2020.
- Burdett K., Shi S., and R. Wright. Pricing and matching with frictions. *Journal of Political Economy*, 109(5):1060–1085, 2001.
- De Sinopoli, F., Meroni C. and C. Pimienta. Strategic stability in Poisson games. *Journal of Economic Theory*, 153(5):46–63, 2014.
- Diamond, P. Aggregate demand management in search equilibrium. *Journal of Political Economy*, 90(3):881–894, 1982.
- Galenianos, M. and P. Kircher. On the game-theoretic foundations of search equilibrium. *International Economic Review*, 53(1):1–21, 2012.
- Godoy, A. and E. Moen. Mixed search. *Mimeo*, 2015.
- Jehiel, P. and L. Lamy. On discrimination in auctions with endogenous entry. *The American Economic Review*, 105(8):2595–2643, 2015.
- Lentz, R. and E. Moen. Competitive or random search. *Mimeo*, 2017.
- Makris, M. Complementarities and macroeconomics: Poisson games. *Games and Economic Behavior*, 62(1):180–189, 2008.
- Makris, M. Private provision of discrete public goods. *Games and Economic Behavior*, 67(1):292–299, 2009.
- Myerson, R. B. Population uncertainty and Poisson games. *International Journal of Game Theory*, 27(3):375–392, 1998.
- Myerson, R. B. Large Poisson games. *Journal of Economic Theory*, 94(1):7–45, 2000.
- Myerson, R. B. Comparison of scoring rules in Poisson voting games. *Journal of Economic Theory*, 103(1):219–251, 2002.
- Moen, E. R. Competitive search equilibrium. *Journal of Political Economy*, 105(2):385–411, 1997.
- Mortensen, D. Job search and labor market analysis. *Handbook of Labor Economics*, North Holland, 1986.
- Norman, P. Matching with frictions and entry with Poisson distributed buyers and sellers. *Mimeo*, 2016.
- Peters, M. Limits of exact equilibria for capacity constrained sellers with costly search. *Journal of Economic Theory*, 95(2):139–168, 2000.
- Pissarides, C. Short-run equilibrium dynamics of unemployment, vacancies, and real wages. *American Economic Review*, 75(4): 676–690, 1985.
- Pissarides, C. Equilibrium unemployment theory. *MIT Press*, 2000.

Ritzberger, K. Price competition with population uncertainty. *Mathematical Social Sciences*, 58(2):145–157, 2009.

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