CEIS Research Papers

ISSN 2610-931X

CEIS Tor Vergata

RESEARCH PAPER SERIES

Vol. 19, Issue 5, No. 515 – October 2021

Efficient Nonparametric Estimation of Generalized Autocovariances

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Abstract

This paper provides a necessary and sufficient condition for asymptotic efficiency of a nonparametric estimator of the generalized autocovariance function of a stationary random process. The generalized autocovariance function is the inverse Fourier transform of a power transformation of the spectral density and encompasses the traditional and inverse autocovariance functions as particular cases. A nonparametric estimator is based on the inverse discrete Fourier transform of the power transformation of the pooled periodogram. The general result on the asymptotic efficiency is then applied to the class of Gaussian stationary ARMA processes and its implications are discussed. Finally, we illustrate that for a class of contrast functionals and spectral densities, the minimum contrast estimator of the spectral density satisfies a Yule-Walker system of equations in the generalized autocovariance estimator.

Keywords: Cramér-Rao lower bound; Frequency Domain; Minimum Contrast Estimation; Periodogram.

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1 Introduction

The autocovariance function and its Fourier transform, the spectral density function, characterise the temporal dependence structure of a stationary stochastic process, and are of fundamental importance in time series analysis and prediction. For Gaussian stationary processes they provide, along with the mean, a complete characterization of the probability distribution of the process as well as the basic ingredients for optimal (minimum mean square) prediction, based on time series observations.

The autocovariance function is estimated nonparametrically by the sample autocovariance function. This estimator has a long tradition in time series analysis, and its properties are demonstrated and discussed in time series textbooks, such as, for instance, Brockwell and Davis (1991, ch. 7), where it is shown that under regularity conditions it has an asymptotically normal distribution and that the elements of the asymptotic covariance matrix are given by the celebrated Bartlett's formula.

The literature has further addressed the important question as to what classes of parametric linear processes admit the sample autocovariance as an asymptotically efficient estimator, i.e., an estimator whose variance achieves the Cramèr-Rao lower bound.

This issue has been investigated by Porat (1987) for Gaussian autoregressive (AR) moving average (MA) mixed processes, based on state-space representations and matrix Lyapunov equation theory. For Gaussian ARMA(r, q) processes with $r \geq q$ the sample autocovariances are asymptotically efficient only in a restricted number of cases, while if $q > r$ none of the sample autocovariances is asymptotically efficient. See also Walker (1995) for an alternative derivation of this result. The result implies that the variance and the first r autocovariances of a pure AR(r) process are efficiently estimated by the sample autocovariances, while for a pure MA process none of the autocovariances is asymptotically efficient.

Kakizawa and Taniguchi (1994) derived in the frequency domain a necessary and sufficient condition for asymptotic efficiency of the sample autocovariances that applies to the more general class of Gaussian stationary processes. Kakizawa (1999) extended the previous results to the case of vector processes. Boshnakov (2005) studied the efficiency of the sample autocovariances for processes obtained by a finite linear transformation of a pure autoregressive process.

The generalized autocovariance (GACV) function was defined in Proietti and Luati (2015) as the inverse Fourier transform of the p-th power of the spectral density function. It encompasses the traditional autocovariance function ($p = 1$) and the inverse autocovariance function (Cleveland, 1972), which is the sequence of the coefficients of the Fourier expansion of the inverse spectrum ($p = -1$).

The aim of this paper is to study asymptotic efficiency of the nonparametric estimator of the GACV considered in Proietti and Luati (2015). Following Hannan and Nicholls (1977) and Luati et al. (2012), the estimator is based on the powers of the pooled periodogram over m non-overlapping consecutive frequencies, where m is the pooling parameter. Proietti and Luati (2015) established consistency and asymptotic normality of the estimator.

We establish a necessary and sufficient condition for asymptotic efficiency in terms of the spectral density and its derivatives for general Gaussian stationary processes, which nests as a particular case the result of Kakizawa and Taniguchi (1994), which holds for $p = 1$. The results also show that the the nonparametric estimator achieves the Cramèr-Rao lower bound as $m \to \infty$ for $p = -1$, i.e. it estimates efficiently the first q inverse autocovariances when the true generating process is pure MA(q), thereby complementing the results by Bhansali (1980) and Battaglia (1988). The inverse autocovariance function is useful in interpolation problems and for the identification of ARMA models.

After characterizing a class of processes for which the nonparametric estimator is fully efficient, we consider the case when the process is Gaussian ARMA(r, q), in which case some numerical example illustrate the rate of convergence to the Cramèr-Rao bound. The results obtained include, as a special case, the results for the sample autocovariance function by and Porat (1987) and Kakizawa and Taniguchi (1994).

Finally, we illustrate that for a class of contrast functionals and spectral densities, the minimum contrast estimator of the spectral density satisfies a Yule-Walker system of equations in the generalized autocovariance estimator.

The paper is organized as follows. Section 2 states the main assumptions concerning the generating process, recalls the definition of the estimator of the GACV and the Cramér-Rao lower bound. Section 3 contains the main result in the paper, establishing a necessary and sufficient condition for asymptotic efficiency of the estimator, and discussing its positioning in the literaure. The asymptotic efficiency when the series is generated by a Gaussian ARMA processes is discussed in section 4, along with some numerical illustrations. Section 5 discusses the the nonparametric GACV estimator as a minimum contrast estimator (Taniguchi, 1987). Section 6 concludes the paper. Proofs are deferred to the Appendix.

2 Basic definitions and assumptions

Let $\{X_t\}_{t \in T}$, with $T \in \mathbb{N}$ a discrete time set, denote a zero mean stationary Gaussian process with autocovariance function $\gamma_k = \mathrm{E}(X_t X_{t-k}), k \in \mathbb{Z}$, and spectral density $f_\theta(\omega) = \sum_{k=-\infty}^{\infty} \gamma_k e^{-\imath \omega k}$, $\omega \in [-\pi, \pi]$, both depending on an $s \times 1$ vector of parameters $\boldsymbol{\theta} = (\theta_1, \dots, \theta_s)' \in \mathbb{R}^s$.

For $p \in \mathbb{R}$ we define the generalized autocovariances, denoted γ_{pk}, as the sequence of Fourier coefficients of $[2\pi f_\theta(\omega)]^p$, i.e.,

$$[2\pi f_\theta(\omega)]^p = \sum_{k=-\infty}^{\infty} \gamma_{pk} e^{-\imath \omega k},$$

or, equivalently,

$$\gamma_{pk} = \frac{1}{2\pi} \int_{-\pi}^{\pi} [2\pi f_\theta(\omega)]^p \cos(k\omega)\, d\omega. \tag{1}$$

Obviously, $\gamma_{1k} = \gamma_k$, while $\gamma_{-1,k}$ is the inverse autocovariance function, see Cleveland (1972) and Battaglia (1983).

Throughout the paper we make the following assumptions.

Assumption 1. There exist two positive constants $\underline{c}$ and $\overline{c}$ such that
$0 < \underline{c} \le f_\theta(\omega) \le \overline{c} < \infty$, for $\omega \in [-\pi, \pi]$.

Assumption 2. The generalized autocovariances and their partial derivatives, $\partial \gamma_{pk}/\partial \theta_j$, $j = 1, \dots, s$, satisfy the summability conditions $\sum_{k=1}^{\infty} k|\gamma_{pk}| < \infty$, $\sum_{k=1}^{\infty} k|\partial \gamma_{pk}/\partial \theta_j| < \infty$.

Assumption 3. The $s \times s$ matrix

$$\frac{1}{4\pi} \int_{-\pi}^{\pi} \frac{\partial f_\theta(\omega)}{\partial \boldsymbol{\theta}} \frac{\partial f_\theta(\omega)}{\partial \boldsymbol{\theta}'} \frac{d\omega}{f_\theta^2(\omega)}$$

is positive definite.

The first assumption restricts attention to short memory processes, ruling out long memory and non-invertible models, see, e.g., Hassler (2018). Assumption 2 implies that $\int_{-\pi}^{\pi} [f_\theta(\omega)]^p \, d\omega < \infty$ and $f_\theta(\omega)$ is differentiable with respect to θ_j, and $\partial f_\theta(\omega)/\partial \theta_j$ is continuous and differentiable with respect to ω, with continuous derivative.

Given a time series of N consecutive observations, $\{x_t, t = 1, 2, \dots, N\}$, and their sample mean $\bar{x}_N = 1/N \sum_{t=1}^{N} x_t$, we define the periodogram

$$I(\omega_j) = \frac{1}{2\pi N} \left| \sum_{t=1}^{N} (x_t - \bar{x}_N) \exp(-i\omega_j t) \right|^2,$$

where ω_j is the Fourier frequency $\omega_j = \frac{2\pi j}{N} \in (0, \pi)$, $1 \leq j \leq \lfloor \frac{N-1}{2} \rfloor$, and $\lfloor \cdot \rfloor$ denotes the integer part of the argument.

Based upon Hannan and Nicholls (1977) and Luati et al. (2012), Proietti and Luati (2015) proposed the following nonparametric estimator of the generalized autocovariances based on the inverse discrete Fourier transform of the p-th power of the corrected pooled periodogram,

$$\hat{\gamma}_{pk} = \frac{1}{M} \sum_{j=0}^{M-1} Y_j^{(p)} \cos(\bar{\omega}_j k), \tag{2}$$

where $M = \lfloor \frac{N-1}{2m} \rfloor$ and

$$Y_j^{(p)} = (2\pi \bar{I}_j)^p \frac{\Gamma(m)}{\Gamma(m+p)},$$

where

$$\bar{I}_j = \sum_{l=1}^{m} I(\omega_{jm+l})$$

is the pooled periodogram over $m \geq 1$ nonoverlapping consecutive frequencies and $\bar{\omega}_j = \omega_{jm+(m+1)/2}$ are mid range frequencies; m is the pooling parameter.

Some constraints on m ensure the existence of the second moment of the pth power of a Gamma random variable (Proietti and Luati, 2015): in particular, we need $p > -m/2$.

Let $\boldsymbol{\gamma}_p = [\gamma_{p0}, \gamma_{p1}, \dots, \gamma_{pK}]'$ be the vector of the generalized autocovariance functions up to lag K and $\hat{\boldsymbol{\gamma}}_p = [\hat{\gamma}_{p0}, \hat{\gamma}_{p1}, \dots, \hat{\gamma}_{pK}]'$ the corresponding estimator. Under the stated assumptions and additional assumptions on m and on the coefficients of the Wold representation of the process, Proietti and Luati (2015) showed that:

$$\sqrt{N}(\hat{\boldsymbol{\gamma}}_p - \boldsymbol{\gamma}_p) \underset{d}{\rightarrow} N(\mathbf{0}, \boldsymbol{V}), \tag{3}$$

where $\boldsymbol{V} = \{v_{kl}, k, l = 1, \ldots, K\}$, with

$$v_{kl} = m\left(C(m;p,p) - 1\right) \frac{1}{\pi} \int_{-\pi}^{\pi} [2\pi f_\theta(\omega)]^{2p} \cos(\omega k) \cos(\omega l)\, d\omega, \tag{4}$$

where

$$C(m;p,p) = \frac{\Gamma(m+2p)\Gamma(m)}{\Gamma^2(m+p)},$$

and $\Gamma(\cdot)$ is the Gamma function.

Definition 1. The GACV estimator $\hat{\gamma}_{kp}$ in (2) is efficient if its asymptotic variance, v_{kk} converges to the *Cramér-Rao lower bound*

$$CRB\{\hat{\gamma}_{pk}\} = \frac{\partial \gamma_{pk}}{\partial \boldsymbol{\theta}'} \Im_N^{-1}(\boldsymbol{\theta}) \frac{\partial \gamma_{pk}}{\partial \boldsymbol{\theta}}, \tag{5}$$

with

$$\frac{\partial \gamma_{pk}}{\partial \boldsymbol{\theta}} = (2\pi)^{p-1} \int_{-\pi}^{\pi} \frac{\partial [f_\theta(\omega)]^p}{\partial \boldsymbol{\theta}} \cos(k\omega)\, d\omega,$$

and $\Im_N(\boldsymbol{\theta})$ is the Fisher information matrix associated with $X_1, \ldots, X_N$,

$$\Im_N(\boldsymbol{\theta}) = \frac{1}{4\pi} \int_{-\pi}^{\pi} \frac{\partial f_\theta(\omega)}{\partial \boldsymbol{\theta}} \frac{\partial f_\theta(\omega)}{\partial \boldsymbol{\theta}'} \frac{1}{f_\theta^2(\omega)}\, d\omega.$$

Under assumptions $1-3$, by (3) and (4), $v_{kk} \geq CRB\{\hat{\gamma}_{pk}\}$ gives the following inequality:

$$\frac{m(C(m;p,p)-1)}{p^2} \int_{-\pi}^{\pi} [f_\theta(\omega)]^{2p} \cos^2(\omega k)\, d\omega \geq$$
$$\left\{ \int_{-\pi}^{\pi} [f_\theta(\omega)]^p \frac{\partial \ln f_\theta(\omega)}{\partial \boldsymbol{\theta}'} \cos(k\omega)\, d\omega \right\}$$
$$\left\{ \int_{-\pi}^{\pi} \frac{\partial \ln f_\theta(\omega)}{\partial \boldsymbol{\theta}} \frac{\partial \ln f_\theta(\omega)}{\partial \boldsymbol{\theta}'}\, d\omega \right\}^{-1} \left\{ \int_{-\pi}^{\pi} [f_\theta(\omega)]^p \frac{\partial \ln f_\theta(\omega)}{\partial \boldsymbol{\theta}'} \cos(k\omega)\, d\omega \right\}'. \tag{6}$$

3 Asymptotic efficiency of the estimator of the GACV

The nonparametric estimator $\hat{\gamma}_{pk}$ of the generalized autocovariance function γ_{pk} is asymptotically efficient if its asymptotic variance attains the Cramér-Rao bound, that is if, in (6), equality holds. This requires a condition on the spectral density of the process and on the limit behaviour of the quantity $m(C(m;p,p)-1)/p^2$, that will be stated in the following theorem.

Theorem 1. *Suppose that assumptions 1-3 are satisfied and that m and M are large enough for asymptotics and $\frac{m}{M}$ is small enough for f to be constant over frequency intervals of length $\frac{2\pi m}{M}$ and $m(C(m,p,p)-1) \to p^2$. Then, $\hat{\gamma}_{pk}$ is asymptotically efficient if and only if there exists an s-dimensional vector $\mathbf{c}$, independent of ω, such that:*

$$[f_\theta(\omega)]^{p+1} \cos(k\omega) + \mathbf{c}' \frac{\partial f_\theta(\omega)}{\partial \boldsymbol{\theta}} = 0, \tag{7}$$

Proof. Proof: see Appendix A. □

The above condition can also be stated as

$$[f_\theta(\omega)]^p \cos(k\omega) + \mathbf{c}' \frac{\partial \ln f_\theta(\omega)}{\partial \boldsymbol{\theta}} = 0.$$

The proof of Theorem 1 is based on a matrix-based integral inequality from Kakizawa and Taniguchi (1994), generalizing the Cauchy-Schwarz inequality and Kholevo's inequality (Kholevo, 1969).

Theorem 1 provides a necessary and sufficient condition for asymptotic efficiency of $\hat{\gamma}_{pk}$ which is valid for general Gaussian stationary processes. It is expressed in terms of the spectral density function, which makes it easy to check for various models. This result embodies in a single equation the condition for asymptotic efficiency of the sample autocovariance function ($p = 1$), of the estimator $\hat{\gamma}_{-1,k}$ of the inverse autocovariance function ($p = -1$), which at lag $k = 0$ provides the inverse of the interpolation error variance, and of the estimator $\hat{\gamma}_{pk}$ for general real powers p.

Corollary 1. *Consider the process with spectral density function* $f_\theta(\omega) = \frac{1}{2\pi}\left[\frac{1}{\theta(\omega)}\right]^{\frac{1}{p}}$, *with* $\theta(\omega)$ *the trigonometric polynomial* $\theta(\omega) = \theta_0 + 2\sum_{j=1}^{K} \theta_j \cos(\omega j)$, *so that* $\frac{\partial \theta(\omega)}{\partial \boldsymbol{\theta}} = \boldsymbol{q}(\omega) = [1, 2\cos(\omega), 2\cos(2\omega), \ldots, 2\cos(\omega K)]'$. *Then,*

$$\frac{\partial f_\theta(\omega)}{\partial \boldsymbol{\theta}} = -(2\pi)^p \frac{1}{p} [f_\theta(\omega)]^{p+1} \boldsymbol{q}(\omega).$$

Condition (7) *in Theorem 1 becomes*

$$[f_\theta(\omega)]^{p+1} \left\{ \cos(k\omega) - \frac{(2\pi)^p}{p} \mathbf{c}' \boldsymbol{q}(\omega) \right\} = 0,$$

which is satisfied if $\boldsymbol{c} = \left[0, 0, \ldots, \frac{p}{2(2\pi)^p}, 0, \ldots, 0\right]'$. *This implies that for* $p = -1$ *the process is moving-average of order* K *and the first* K *inverse autocovariances* $\boldsymbol{\gamma}_{-1,K} = [\gamma_{-1,1}, \ldots, \gamma_{-1,K}]'$ *and* $\gamma_{-1,0}$ *can be efficiently estimated as* $N \to \infty$ *by the estimator of the GACV* $\hat{\boldsymbol{\gamma}}_{-1,K}$ *with large* m.

Remark 1. The estimator (2) can be viewed in the wider context of estimation of functionals of the spectral density, which are related to many important quantities in stationary time series. Setting $m = 1$, for $p > 0$, Y_j^p is the inverse Laplace transform of $[2\pi f(\omega_j)]^{-(p+1)}$ evaluated at $2\pi I(\omega_j)$, proposed by Taniguchi (1980) for estimating $[2\pi f(\omega_j)]^p$. Asymptotic efficiency of this estimator is studied in Taniguchi (1981), who establishes that this estimator is asymptotically efficient if $p = 1$ and the spectral density is constant over $[-\pi, \pi]$. The nonparametric estimator of $\hat{\gamma}_{pk}$ further generalizes these results to any real power transform, including negative p. Furthermore, the introduction of the pooling parameter m allows asymptotically efficient estimates also for $p \neq 1$.

Remark 2. By setting the power p and the pooling parameter m to 1, inequality (6) reduces to the asymptotic Cramér-Rao inequality for the sample estimator of the autocovariance function analysed by Kakizawa and Taniguchi (1994). Note also that for $p = 1$, by the properties of the Gamma function, the constant $m(C(m; p, p) - 1)$ does not depend on the pooling parameter m: $m(C(m; , 1, 1) - 1) = 1$. Hence, if we consider estimation of the traditional autocovariance function, the asymptotic variance of the nonparametric estimator $\hat{\gamma}_{1k}$ does not depend on the pooling parameter m. Indeed, $\hat{\gamma}_{1k}$ is the Riemannian sum approximation over the Fourier frequencies of the sample autocovariance at lag k, denoted by $\tilde{\gamma}_k$:

$$\lim_{N\to\infty} \frac{1}{\lfloor (N-1)/2 \rfloor} \sum_{j=1}^{\lfloor (N-1)/2 \rfloor} 2\pi I(\omega_j) \cos(\omega_j k) = \int_{-\pi}^{\pi} I(\omega) \cos(\omega k)\, d\omega = \tilde{\gamma}_k,$$

with $I(\omega) = \frac{1}{2\pi} \sum_{|h|<N} \tilde{\gamma}_h \cos(\omega h)$. Hence $\lim_{N\to\infty} \hat{\gamma}_{1k} = \tilde{\gamma}_k$, and their asymptotic variances, as $N \to \infty$, are equivalent. As a matter of fact, by setting $p = 1$, Theorem 1 provides the condition for asymptotic efficiency of the sample autocovariances by Kakizawa and Taniguchi (1994).

4 Numerical illustrations

Some specific cases of ARMA processes are considered and the performance of the nonparametric estimator of interest is related to the Cramér-Rao lower bound.

Let us consider a stationary AR(1) process $\{X_t\}_{t\in T}$, $|\phi| < 1$, with spectral density function

$$f_\theta(\omega) = \frac{\sigma^2}{2\pi} \frac{1}{1 - 2\phi\cos\omega + \phi^2}$$

The parameter vector is $\boldsymbol{\theta} = (\phi, \sigma^2)'$. We denote the asymptotic variance of $\hat{\gamma}_{pk}$ by

$$AV\{\hat{\gamma}_{pk}\} = m(C(m; p, p) - 1)\frac{\sigma^{2p}}{\pi} \int_{-\pi}^{\pi} \left(\frac{1}{1 - 2\phi\cos\omega + \phi^2}\right)^{2p} \cos^2(k\omega)\, d\omega.$$

Table 1 refers to a stationary AR(1) process with parameters $\phi = 0.8$ and $\sigma^2 = 1$. It displays the values of $AV\{\hat{\gamma}_{pk}\}$, $CRB\{\hat{\gamma}_{pk}\}$ and their ratio for different values of m and p. Recall that for Gaussian processes it must be $m > -2p$. Values greater than one measure the inefficiency of the estimator (2). As we know, the sample autocovariance function $\hat{\gamma}_{1k}$ is asymptotically efficient for $k = 0, 1$. Except for this case, exact equality between the asymptotic variance and Cramér-Rao bound of $\hat{\gamma}_{pk}$ never holds, but it is approximated as m increases.

Table 1: Asymptotic efficiency of $\hat{\gamma}_{pk}$ for an AR(1) model with $\phi = 0.8$ and $\sigma^2 = 1$.

k	p	m	AV/CRB	k	AV/CRB	k	
		1	1.36		1.38		
1	2	2	1.26	2	1.29		
		30	1.10		1.12		
4	2	30	1.21	5	1.28	7	1.53
		50	1.20		1.27		1.52

We focus on the case when $p = 2$, which is of interest when applied to autoregressive functions as positive powers emphasise peaks in the spectral density function. Examples of spectral density functions showing a peak are those associated with processes that exhibit cyclical behavior, often represented by AR(2) or ARMA(2,1) processes. For positive values of p, no constraint on the pooling parameter is required, so we let $m = 1, 2, 30, 50$ for $k = 1, 2, 4, 5, 7$, to show the effect of pooling. As it is shown in Table 1, pooling has a positive effect on the asymptotic efficiency of the estimator, as a value of $m = 30$ reduces its asymptotic variance by 18.7%.

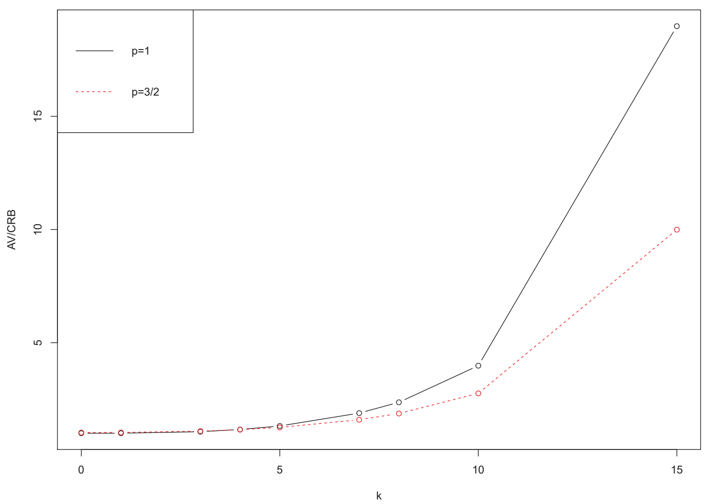

Figure 1: Plot of AV/CRB against k for $p = 1.5$ (dotted line) and $p = 1$ (line), with $m = 30$, for an AR(1) process with $\sigma^2 = 1, \phi = 0.8$.

A comparison with the sample autocovariance evidences that the ratios AV/CRB relative to $\hat{\gamma}_{pk}$ and $\hat{\gamma}_{1k}$ both increase as k increases, and the difference between them also becomes larger in favour of the estimator of the GACV for several values of p. This emerges froom the plot, in Figure 1, of the ratio AV/CRB against k, where the red dashed line refers to the estimator of the GACV for $p = 3/2$ and the black solid line refers to the sample autocorerlation.

We now move to estimation of the inverse autocorrelation when moving average processes or ARMA(r, q) processes are considered, with $q > r$. Table 2 reports the asymptotic efficiency of $\hat{\gamma}_{pk}$ for $p = -1$: the theoretical results are confirmed and very good efficiency results are obtained for increasing m, when, in general, the results for $p = 1$ are inefficient. Indeed, the ratio AV/CRB for the estimator of the inverse autocovariance function ($p = -1$) with $k = 1$ is 1.08 for $m = 25$ and 1.04 with $m = 50$, very close to unity, while the same ratio for for $p = 1$ and $k = 1$ is equal to 2.47.

Table 2: Asymptotic efficiency of $\hat{\gamma}_{pk}$ for an MA(1) model with $\theta = 0.7, \sigma^2 = 1$.

k	p	m	AV/CRB	k	AV/CRB	k	AV/CRB
		3	3.00		3.18		3.72
		5	1.66		1.76		2.06
1	-1	10	1.25	2	1.32	3	1.55
		25	1.08		1.15		1.34
		50	1.04		1.10		1.29
		3	4.80		6.73		
		5	2.66		3.73		
4	-1	10	2.00	5	2.80		
		25	1.73		2.43		
		50	1.66		2.33		

Table 3 shows a further example concerning an ARMA(1,2) model. In accordance with the previous tables, results get worse for larger k. These results also confirm that asymptotic efficiency of the estimator $\hat{\gamma}_{-1,k}$ of the inverse autocovariance function at each lag k strongly improves as m increases. This is apparent if comparing the ratio AV/CRB when $m = 3$ to that with $m = 50$.

Table 3: Asymptotic efficiency of $\hat{\gamma}_{pk}$ for an ARMA(1,2) model with $\theta_1 = -0.7$, $\theta_2 = 0.1$, $\phi = 0.6$, $\sigma^2 = 1$.

k	p	m	AV/CRB	k	AV/CRB	k	AV/CRB
		3	3.00		3.44		7.33
		5	1.66		1.91		4.07
1	-1	10	1.25	2	1.43	3	3.05
		25	1.08		1.24		2.65
		50	1.04		1.19		2.54
		3	8.18		9.31		
		5	4.54		5.17		
4	-1	10	3.41	5	3.88		
		25	2.96		3.37		
		50	2.84		3.23		

5 Minimum contrast estimation

As in Corollary 1, let us consider the process with spectral density function

$$[2\pi f_\theta(\omega)]^p = [\theta(\omega)]^{-1}, \tag{8}$$

where $\theta(\omega) > 0$ is the trigonometric polynomial $\theta_0 + 2\sum_{k=1}^K \theta_k \cos(\omega k)$. Writing $\theta(\omega) = \theta_0|\phi(e^{-\imath\omega})|^2$, $\phi(e^{-\imath\omega}) = 1 - \sum_{j=1}^K \phi_j e^{-\imath\omega j}$, such that $\theta_k = \theta_0 \sum_{j=1}^{K-k} \phi_j\phi_{j+k}$, and setting $\sigma^2 = \theta_0^{-1}$, it can be seen, by integrating both sides of (8) over $\omega \in [-\pi, \pi]$, that γ_{pk} is the autocovariance function of the AR(K) process $U_t = \sum_{j=1}^s \phi_j U_{t-j} + \sigma\epsilon_t, \epsilon_t \sim$ i.i.d. N$(0,1)$.

Following Taniguchi (1987), let us consider minimum contrast (MC) estimation of the spectral density $f_\theta(\omega)$ using the contrast functional

$$K(z;p) = \ln(z^p) + \frac{1}{z^p},$$

applied to $f_\theta(\omega)/g_N(\omega)$, where $g_N(\omega)$ is the corrected pooled periodogram, $g_N(\omega) = \bar{I}(\omega)\sqrt[p]{\Gamma(m)/\Gamma(m+p)}$ such that $\mathrm{E}\{[g_N(\omega)]^p\} = [f_\theta(\omega)]^p$.

Define

$$Y(\omega) = \frac{1}{2\pi}\sum_{-M+1}^{M-1} \hat{\gamma}_{pk} e^{-\imath\omega k}, \omega \in [-\pi, \pi],$$

so that $\hat{\gamma}_{pk} = \int_{-\pi}^{\pi} Y(\omega)e^{\imath\omega k}d\omega$, and $g_N(\omega) = [Y(\omega)]^{1/p}$.

The MC estimator of $(\phi_1, \ldots, \phi_K, \sigma^2)'$ is the minimizer of

$$\int_{-\pi}^{\pi} K\left(\frac{f_\theta(\omega)}{g_N(\omega)}, p\right) d\omega = \int_{-\pi}^{\pi} \left\{\ln\sigma^2 - \ln|\phi(e^{-\imath\omega})|^2 - \ln Y(\omega) + \frac{1}{\sigma^2}Y(\omega)|\phi(e^{-\imath\omega})|^2\right\} d\omega.$$

The MC estimator of σ^2 is $\hat{\sigma}^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} Y(\omega)|\hat{\phi}(e^{-\imath\omega})|^2 d\omega$. Replacing in the contrast function (and noticing $\int_{-\pi}^{\pi} |\hat{\phi}(e^{-\imath\omega})|^2 d\omega = 0$), the MC estimator of $\boldsymbol{\phi} = (\phi_1, \ldots, \phi_s)'$ is the minimizer of the criterion function

$$Q(\boldsymbol{\phi}) = \int_{-\pi}^{\pi} Y(\omega)|\phi(e^{-\imath\omega})|^2 d\omega.$$

Writing

$$|\phi(e^{-\imath\omega})|^2 = 1 - 2\boldsymbol{\phi}'\mathbf{b}(\omega) + \boldsymbol{\phi}'\mathbf{B}(\omega)\boldsymbol{\phi}$$

where $\mathbf{b}(\omega) = [\cos\omega, \cos(2\omega), \ldots, \cos(\omega K)]'$ and $\mathbf{B}(\omega) = \{\cos(\omega(h-k)), h, k = 1, 2, \ldots, s\}$, differentiating with respect to $\boldsymbol{\phi}$ and setting the derivatives equal to zero yields

$$\frac{\partial Q}{\partial \boldsymbol{\phi}} = \int_{-\pi}^{\pi} Y(\omega)(\mathbf{b}(\omega) - \mathbf{B}(\omega)\boldsymbol{\phi})d\omega \equiv 0,$$

which is the generalized Yule-Walker system of equations:

$$\hat{\gamma}_{pk} = \sum_{j=1}^K \hat{\phi}_j \hat{\gamma}_{p,k-j}, k = 1, 2, \ldots, K.$$

Hence, an asymptotically efficient estimator of $(\boldsymbol{\phi}, \sigma^2)$, and thus of $\boldsymbol{\theta}$, can be obtained by solving a generalized Yule-Walker system based on the GACV estimator (2).

6 Concluding remarks

The paper has established a necessary and sufficient condition in the frequency domain for asymptotic efficiency of the nonparametric estimator of the GACV proposed by Proietti and Luati (2015). The result generalizes the condition for asymptotic efficiency of the sample autocovariances provided by Kakizawa and Taniguchi (1994), and applies to non-parametric estimation of the inverse autocovariance function and of the Fourier coefficients of general power transformations. The condition derived is easy to check for various models.

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A Proof of Theorem 1

We first recall the following Lemma by Kakizawa and Taniguchi (1994), which allows us to compare the asymptotic variance of $\hat{\gamma}_{pk}$ and its Cramèr-Rao lower bound:

Lemma 1. Let $\mathcal{A}(\omega)$ and $\mathcal{B}(\omega)$ be $r \times s, t \times s$ matrices, respectively, and let $g(\omega)$ be a function such that $g(\omega) > 0$ almost everywhere (a.e.) on $[-\pi, \pi]$. If the matrix

$$\left\{ \int_{-\pi}^{\pi} \frac{\mathcal{B}(\omega)\mathcal{B}(\omega)'}{g(\omega)} \, d\omega \right\}^{-1}$$

exists, then

$$\int_{-\pi}^{\pi} \mathcal{A}(\omega)\mathcal{A}(\omega)' g(\omega) \, d\omega \geq \left\{ \int_{-\pi}^{\pi} \mathcal{A}(\omega)\mathcal{B}(\omega)' \, d\omega \right\} \left\{ \int_{-\pi}^{\pi} \frac{\mathcal{B}(\omega)\mathcal{B}(\omega)'}{g(\omega)} \, d\omega \right\}^{-1} \left\{ \int_{-\pi}^{\pi} \mathcal{A}(\omega)\mathcal{B}(\omega)' \, d\omega \right\}'$$

where $\geq$ means the left-hand side minus the right-hand side is positive semi-definite. Here the equality holds if there exists an $r \times t$ matrix $\mathcal{C}$ which is independent of ω such that:

$$g(\omega)\mathcal{A}(\omega) + \mathcal{C}\mathcal{B}(\omega) = 0.$$

Asymptotic efficiency of $\hat{\gamma}_{pk}$ occurs when v_{kk} achieves the CRB on the right hand side of (6). The latter can be rewritten as

$$K \int_{-\pi}^{\pi} \mathcal{A}(\omega)\mathcal{A}(\omega)' g(\omega) \, d\omega \geq$$

$$\left\{ \int_{-\pi}^{\pi} \mathcal{A}(\omega)\mathcal{B}(\omega)' \, d\omega \right\} \left\{ \int_{-\pi}^{\pi} \frac{\mathcal{B}(\omega)\mathcal{B}(\omega)'}{g(\omega)} \, d\omega \right\}^{-1} \left\{ \int_{-\pi}^{\pi} \mathcal{A}(\omega)\mathcal{B}(\omega)' \, d\omega \right\}',$$

with $\mathcal{A}(\omega) = \cos(k\omega)[f_\theta(\omega)]^{p-1}$, $\quad \mathcal{B}(\omega) = \frac{\partial f_\theta(\omega)}{\partial \boldsymbol{\theta}}$, $\quad g(\omega) = f_\theta^2(\omega)$, and $K = \frac{m(C(m;p,p)-1)}{p^2}$.

The attainment of the CRB thus depends also on the term $\frac{m(C(m;p,p)-1)}{p^2}$, involving both the power p and the pooling parameter m.

It is possible to show that $K \to 1$ as $m \to \infty$. We start by using a result about the approximation of a quotient of two Gamma functions, obtained by the use of the Stirling's series (Erdélyi et al., 1954):

$$\frac{\Gamma(z+\alpha)}{\Gamma(z+\beta)} = z^{\alpha-\beta}\left[1 + \frac{(\alpha-\beta)(\alpha+\beta-1)}{2z} + O(|z|^{-2})\right],$$

as $z \to \infty$, where α and β are bounded. By using this approximation we can rewrite

$$\begin{aligned} \frac{\Gamma(m)\Gamma(m+2p)}{[\Gamma(m+p)][\Gamma(m+p)]} &\approx m^{-p}\left[1 + \frac{(-p)(p-1)}{2m}\right] m^p \left[1 + \frac{p(3p-1)}{2m}\right] \\ &= \frac{4m^2+4mp^2-3p^4+4p^3-p^2}{4m^2}, \end{aligned}$$

By a change of variable and the De L'Hôpital theorem we find that $m(C(m;p,p)-1) \to p^2$ as $m \to \infty$.

Hence, in the cases $p = 1$ or $p \neq 1$ and $m \to \infty$, the Cramér-Rao inequality (6) becomes:

$$\int_{-\pi}^{\pi} f_\theta^2(\omega)[f_\theta(\omega)]^{2(p-1)} \cos^2(\omega k)\, d\omega \geq$$
$$\left\{ \int_{-\pi}^{\pi} [f_\theta(\omega)]^{p-1} \frac{\partial f_\theta(\omega)}{\partial \boldsymbol{\theta'}} \cos(k\omega)\, d\omega \right\}$$
$$\left\{ \int_{-\pi}^{\pi} \frac{\partial f_\theta(\omega)}{\partial \boldsymbol{\theta}} \frac{\partial f_\theta(\omega)}{\partial \boldsymbol{\theta'}} \frac{1}{f_\theta^2(\omega)}\, d\omega \right\}^{-1} \left\{ \int_{-\pi}^{\pi} [f_\theta(\omega)]^{p-1} \frac{\partial f_\theta(\omega)}{\partial \boldsymbol{\theta'}} \cos(k\omega)\, d\omega \right\}'.$$

Applying Lemma 1 by setting:

$$\mathcal{A}(\omega) = \cos(k\omega)[f_\theta(\omega)]^{p-1}, \quad \mathcal{B}(\omega) = \frac{\partial f_\theta(\omega)}{\partial \boldsymbol{\theta}}, \quad g(\omega) = f_\theta^2(\omega),$$

proves Theorem 1. $\square$

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