

CEIS Tor Vergata

RESEARCH PAPER SERIES

Vol. 20, Issue 2, No. 534 – March 2022

Dimension Reduction for High Dimensional Vector Autoregressive Models

Gianluca Cubadda and Alain Hecq

Dimension Reduction for High Dimensional Vector Autoregressive Models

Gianluca Cubadda*

Università di Roma "Tor Vergata"

Alain Hecq[†]

Maastricht University

February 19, 2022[‡]

Abstract

This paper aims to decompose a large dimensional vector autoregressive (VAR) model into two components, the first one being generated by a small-scale VAR and the second one being a white noise sequence. Hence, a reduced number of common components generates the entire dynamics of the large system through a VAR structure. This modelling, which we label as the dimension-reducible VAR, extends the common feature approach to high dimensional systems, and it differs from the dynamic factor model in which the idiosyncratic component can also embed a dynamic pattern. We show the conditions under which this decomposition exists. We provide statistical tools to detect its presence in the data and to estimate the parameters of the underlying small-scale VAR model. Based on our methodology, we propose a novel approach to identify the shock that is responsible for most of the common variability at the business cycle frequencies. We evaluate the practical value of the proposed methods by simulations as well as by an empirical application to a large set of US economic variables.

Keywords: Vector autoregressive models, dimension reduction, reduced-rank regression, multivariate autoregressive index model, common features, business cycle shock.

*Università di Roma "Tor Vergata", Dipartimento di Economia e Finanza, Via Columbia 2, 00133 Roma, Italy. Email: gianluca.cubadda@uniroma2.it.

[†]Maastricht University, Department of Quantitative Economics, P.O.Box 616, 6200 MD Maastricht, The Netherlands. Email: a.hecq@maastrichtuniversity.nl.

[‡]Various versions of this paper were presented at the 31th (EC)² Conference on "High dimensional modeling in time series", the 9th Italian Congress of Econometrics and Empirical Economics, the International Association for Applied Econometrics 2021 Annual Conference, the workshop on "Dimensionality Reduction and Inference in High-Dimensional Time Series" at Maastricht University, the 15th International Conference on Computational and Financial Econometrics, the 3rd Italian Workshop of Econometrics and Empirical Economics: "High-dimensional and Multivariate Econometrics: Theory and Practice", and at a seminar at Sichuan University. We thank the participants, as well as Massimiliano Marcellino, Sung K. Ahn and two anonymous referees, for helpful comments and suggestions. The usual disclaimers apply.

1 Introduction

For decades, the vector autoregressive (VAR) model is *de facto* a standard tool for investigating multivariate time series data.¹ In macroeconometrics, VARs are routinely used for forecasting, for extracting co-movements such as the presence of cointegration, to test for Granger causality as well as to perform structural analyses. However, the number of VAR parameters to be estimated increases quadratically with the number of variables and linearly with the number of lags. This quickly compromises the estimation results outside the case of small scale models. There is currently a growing interest for jointly modeling many variables and consequently to look at the feasibility to work with large dimensional VAR models. Indeed, the increase of data availability in economics and finance² is associated with the common belief that using more information with high dimensional econometric and statistical models will improve our understanding of the macroeconomy as well as forecast accuracy (see Boivin and Ng, 2006 for counter examples). Consequently, as the increase of the number of time series jointly considered in VARs cannot be too large compared to a given number of observations, different attempts have been proposed in the literature to curb the curse of dimensionality problem. These methods can be gathered in two categories: dimension reduction approaches on the one hand and regularization techniques on the other hand. We include in the latter group both Bayesian methods (surveyed in Karlsson, 2013, Koop, 2018), although Bayesian techniques are also used to estimate large reduced-rank VARs (see Carriero *et al.*, 2011), and the more recent booming contributions on penalized estimation of sparse VARs (Wilms and Croux, 2016, Hsu *et al.*, 2008, Nicholson *et al.*, 2018, Davis *et al.*, 2016, Smeekes and Wijler, 2018, Kock and Callot, 2015, Hecq *et al.*, 2021). The former group of methods, to which our paper wishes to contribute, includes reduced rank techniques (Reinsel, 1983, Ahn and Reinsel, 1988, Carriero *et al.*, 2011, Cubadda and Hecq, 2011, Cubadda and Hecq, 2021, Bernardini and Cubadda, 2015) and the huge literature on factor models (surveyed in Stock and Watson, 2016, and Lippi, 2018, 2019).

Differently from those contributions on system dimension reductions, we provide a framework where the whole dynamics of the system is due to an underlying small scale VAR model. Following Lam *et al.* (2011) and Lam and Yao (2012), we first decompose the large multivariate time series Y_t into two parts: a linear function of a small scale dynamic component x_t and a static component ε_t that, and this is the key point that makes our approach different from the usual dynamic factor model, is unpredictable from the past. The capital Y_t stresses that we start from a potentially high-dimensional time series process whereas the small x_t stresses that a small number of factors are responsible for the entire dynamics of the system. Then we provide the conditions under which such a dynamic component x_t is generated by a small scale VAR model. Particularly, we show that it is required that the large VAR model of series Y_t is endowed with both the serial correlation common feature (Engle and Kozicki, 1993) and an index structure (Reinsel, 1983) in order to ensure that the dynamic component x_t follows a VAR model. Hence, we provide a link between the factor modeling for high-dimensional time series and the reduced-rank VAR approach, which, to the best of our knowledge, was not noted before. This bridge allows us to unravel common cyclical features and to impose their presence in large VARs.

Obviously, the decomposition that we consider might not exist. Based on the eigenanalysis proposed in Lam *et al.* (2011) and Lam and Yao (2012), we provide statistical tools to verify whether there exists in

¹It is usual and correct to quote Sims (1980) for his fundamental contribution to the VAR literature. An earlier reference is Quenouille (1957).

²Both in terms of the number of series that are easily available and in the use of different sampling frequencies (e.g. mixed frequency VARs, see Goetz *et al.*, 2016).

series Y_t a dynamic component x_t that is generated by such a small scale VAR model and to estimate the associated parameters. If this is the case, the forecasts of x_t can be used to predict the future realizations of the large dimensional system Y_t and structural shocks may be recovered from the reduced form errors of the dynamic component only.

Our contribution can be related to the early literature on the analysis of linear transformations of vector autoregressive moving average (VARMA) processes, see, *inter alia*, Kohn (1982), and Lütkepohl (1984a, 1984b). These previous contributions established that non-singular linear transformations of a VARMA process still have a VARMA structure, and if some restrictions on the relationships among variables apply, the VARMA model for the linear transformation can be parsimonious enough to be used in empirical applications. However, at the representation theory level the aim of our analysis is rather to find the conditions that allow to model a large dimensional VAR process through a small scale VAR of the same order without any information loss. To the best of our knowledge, such issue has not been tackled so far. So doing we also provide the basis for further extending the link between the final equation representation and VAR models with reduced rank restrictions (Cubadda *et al.*, 2009).

The rest of the paper is organized as follows. Section 2 presents the main results on the model representation as well as the restrictions that our new modelling entails. We observe that, so far, the reduced-rank VAR and multivariate autoregressive index model have been considered separately in the literature. We find out that the combination of the two models allows for the important dimension reduction in large VARs that we seek. We show that information criteria can be used to determine the dimension of a small VAR within a large dimensional system and we provide estimators for its parameters. Section 3 conducts an extensive Monte Carlo study to evaluate the finite sample properties of the proposed tools. Section 4 provides an empirical application a large set of macroeconomic and financial time series to illustrate the practical value of our approach. Based on our methodology, we propose a novel approach to identify the shock that is responsible for most of the common volatility at the business cycle frequency band. Finally, Section 5 concludes.

2 Theory

This section starts by presenting the derivation of the proposed modelling, then it discusses the statistical inference in a large dimensional framework.

2.1 Model representation

Let us assume that the n -vector time series Y_t is generated by the following second-order stationary VAR(p) model

$$Y_t = \sum_{j=1}^p \Phi_j Y_{t-j} + u_t, \quad (1)$$

where $t = 1, \dots, T$, Φ_j is an $n \times n$ matrix for $j = 1, \dots, p$ with $\Phi_p \neq 0$ such that the roots of $\det(I_n - \sum_{j=1}^p \Phi_j z^j)$ lie outside the unit circle; u_t is an n -vector of errors such that $E(u_t) = 0$, $E(u_t u_t') = \Sigma_u$ is a finite and positive definite matrix, $E(u_t | F_{t-1}) = 0$ and F_t is the natural filtration of the process Y_t . For the sake of simplicity, we assume that deterministic elements are absent (or that the series have been demeaned or detrended first).

Assumption 1 For $\Phi = [\Phi_1, \dots, \Phi_p]'$ it holds that $\Phi' = \bar{A}\bar{\Omega}'$, where \bar{A} is a full rank $n \times r$ ($r < n$) matrix and $\bar{\Omega} = [\bar{\omega}'_1, \dots, \bar{\omega}'_p]'$ is a full rank $np \times r$ matrix. Since we can always use the equivalent factorization $\Phi' = A\Omega'$,

where $A = \bar{A}(\bar{A}'\bar{A})^{-1/2}$ and $\Omega = \bar{\Omega}(\bar{A}'\bar{A})^{1/2}$, we assume without loss of generality that A is a matrix with orthogonal columns, namely $A'A = I_r$.

Assumption 1 is popularly known in time series econometrics as the serial correlation common feature (Engle and Kozicki, 1993). It was extensively studied in connection with cointegration (see, *inter alia*, Vahid and Engle, 1993, Ahn, 1997, Cubadda and Hecq, 2001, Hecq *et al.* 2006, Cubadda, 2007, and Athanasopoulos *et al.*, 2011). Moreover, it implies that the marginal processes of series Y_t follow parsimonious univariate models, thus solving the so-called autoregressivity paradox (Cubadda *et al.*, 2009). See Centoni and Cubadda (2015) and Cubadda and Hecq (2021) for recent surveys. In the analysis that follows, we focus on the case where n is large, virtually with a similar magnitude as the sample size T , whereas r is small compared to T .

We start by noting that under Assumption 1 we can use the identity

$$AA' + A_{\perp}A'_{\perp} = I_n \quad (2)$$

in order to decompose series Y_t as

$$Y_t = Ax_t + \varepsilon_t, \quad (3)$$

where $x_t = A'Y_t$, $\varepsilon_t = A_{\perp}A'_{\perp}u_t$ and A_{\perp} is a full-rank $n \times (n-r)$ matrix such that $A'_{\perp}A = 0$ and $A'_{\perp}A_{\perp} = I_{n-r}$.

Following Lam *et al.* (2011) and Lam and Yao (2012), we call x_t and ε_t respectively the dynamic and the static component of series Y_t . Indeed, we have for the disturbances ε_t

$$E(\varepsilon_t | F_{t-1}) = A'_{\perp}A_{\perp}E(u_t | F_{t-1}) = 0$$

from which it follows

$$E(Y_{t+k} | F_t) = AE(x_{t+k} | F_t), \quad (4)$$

where k , the forecast horizon, is any positive integer.

Remark 1 Note that the assumption that the matrix Ω as defined in Assumption 1 has full column rank is equivalent to require that no linear combinations of x_t are innovations w.r.t the past. Hence, decomposition (3) allows us to disentangle the latent autocorrelated component x_t , whose dimension cannot be further reduced, from the unpredictable component ε_t .

Remark 2 Representation (3) has some analogies with a factor model but there are substantial differences. First, the static component ε_t differs from idiosyncratic shocks in approximate factor models (Chamberlain and Rothschild, 1983) in that the former is singular being driven by the $(n-r)$ shocks $A'_{\perp}u_t$ whereas the latter are only mildly cross-correlated.³ Second, the dynamic component x_t drives all the dynamics of the system whereas the idiosyncratic components can also embed a dynamic pattern. Third, in most of the factor literature it is assumed that the factors and the idiosyncratic components are uncorrelated at any lead and lag whereas in Equation (3) we have that $E(\varepsilon_{t+k}x'_t) = 0$ only for $k > 0$.

Lam *et al.* (2011) and Lam and Yao (2012) showed how to consistently estimate both r and A (or, more formally, an $n \times r$ matrix that lies in the space generated by the columns of A) under assumptions that are compatible with those in Assumption 1 even when the dimension n diverges.⁴ Their method is simple to

³This difference has important implications on the inferential properties of the two methodologies. Indeed, Lam *et al.* (2011) show by simulations that their approach outperforms principal component methods when strong cross-correlations exist in the noise.

⁴Notice that the aforementioned contributors assume that $E(\varepsilon_t x'_t) = 0$. We tackle with this issue in Remark 4.

perform since it is based on the eigenanalysis of the sum of the squared autocovariance matrices of series Y_t . Having obtained an estimator, say \hat{A} , it is then possible to get the dynamic and static components in (3) respectively as

$$\hat{x}_t = \hat{A}'Y_t$$

and

$$\hat{\varepsilon}_t = (I - \hat{A}\hat{A}')Y_t.$$

However, in order to forecast series Y_t , as well as to perform structural analysis, estimating the loading matrix A is not enough. In a dimension reduction perspective, we further look at the conditions under which the dynamic component x_t is generated by a small-scale VAR(p) model. To the best of our knowledge, such goal has not been pursued so far.

We start by noting that under Assumption 1 we can rewrite model (1) as follows

$$Y_t = \sum_{j=1}^p A\omega'_j Y_{t-j} + u_t, \quad (5)$$

which is popularly known as the reduced-rank VAR model (RRVAR) and it was extensively analyzed, *inter alia*, by Velu *et al.* (1986) and Ahn and Reinsel (1988). We have already emphasized that restrictions such as (5) are at the heart of the serial correlation common feature literature.

Second, pre-multiplying both sides of Equation (5) by A' we get

$$x_t = \sum_{j=1}^p \omega'_j Y_{t-j} + \xi_t, \quad (6)$$

where $\xi_t = A'u_t$. However, it is important to see that Equation (6) does not yet provide a small-scale model for series Y_t . Indeed, Ω being a $np \times r$ matrix with elements ω in (6), the number of parameters still grows proportionally with n , although not with n^2 as in the unrestricted VAR.

Finally, we insert Equation (3) into (6) such that we obtain

$$x_t = \sum_{j=1}^p \omega'_j A x_{t-j} + \sum_{j=1}^p \omega'_j \varepsilon_{t-j} + \xi_t,$$

which allows us to derive the condition under which the dynamic component x_t is generated by a VAR(p) process as follows:

Assumption 2 *For any $j = 1, \dots, p$ it holds that $\omega_j \in \text{Sp}(A)$, where $\text{Sp}(A)$ indicates the space generated by the columns of A . Notice that this is equivalent to require that $\omega_j = A\alpha'_j$, where α_j is a $r \times r$ matrix.*

Indeed, under Assumption 2 we have that $\omega'_j A x_{t-j} = \alpha_j x_{t-j}$ and $\omega'_j \varepsilon_{t-j} = 0$ for $j = 1, \dots, p$, hence the data generating process of the dynamic component x_t boils down to

$$x_t = \sum_{j=1}^p \alpha_j x_{t-j} + \xi_t. \quad (7)$$

The intuition behind the algebra is that Assumption 2 requires that the lags of the same linear combinations of Y_t that are unpredictable from the past are also irrelevant predictors of the dynamic component x_t .

Remarkably, the RRVAR model (5) of series Y_t can be rewritten as

$$Y_t = \sum_{j=1}^p A\alpha_j \underbrace{A'Y_{t-j}}_{x_{t-j}} + u_t. \quad (8)$$

Model (8) is interesting since it combines the features of the RRVAR model with those of the multivariate autoregressive index (MAI) model proposed by Reinsel (1983). Recently, there has been a renewed interest in the MAI because it allows to rewrite the VAR in a similar way as the popular dynamic factor model, see *inter alia* Carriero *et al.* (2016), Cubadda *et al.* (2017), and Cubadda and Guardabascio (2019). So far, the RRVAR and the MAI have been considered separately in the literature whereas Assumption 2 reveals that the combination of the two models allows for an important dimension reduction in large VARs. In what follows, we call model (8) as the dimension-reducible VAR model (DRVAR).

Remark 3 *Another popular approach in econometrics that aims at exploiting the information of large dimensional time series is the factor augmented VAR (FAVAR) as originally proposed by Bernanke et al. (2005). In such modelling, first some unobserved factors are extracted from an high dimensional time series, then it is assumed that these factors along with a small set of key observed variables jointly follow a small-scale VAR model. Under the assumption that the joint data generating process (DGP) of the observed variables Y_t is (1), it follows that the FAVAR is a restricted case of the DRVAR with*

$$A = \begin{bmatrix} I_m & 0_{(n-m) \times m} \\ 0_{m \times (r-m)} & B_{(n-m) \times (r-m)} \end{bmatrix} \quad (9)$$

such that $x_t = [y_t', w_t']'$, where $y_t = [y_{1,t}, \dots, y_{m,t}]'$ is the m -vector ($m \leq r$) of the key observed variables and $w_t = B'[y_{m+1,t}, \dots, y_{n,t}]'$ is an $(r-m)$ -vector of unobserved factors. Whether restrictions (9) are generally valid is an empirical issue. However, we remark that a different choice of the key variables would induce a different set of restrictions on the A matrix. Since such choice is of course arbitrary, the FAVAR reveals to have a rather *ad hoc* structure.

In order to perform structural analysis through the DRVAR, one way to go is inverting the polynomial VAR coefficient matrix in Equation (8) to obtain the Wold representation of series Y_t . Here we offer an alternative route. If we invert the polynomial coefficient matrix in Equation (7) and insert the Wold representation of the dynamic components x_t in decomposition (3) we get

$$Y_t = A\gamma(L)\xi_t + \varepsilon_t, \quad (10)$$

where $\gamma(L)^{-1} = I_n - \sum_{j=1}^p \alpha_j L^j$. Finally, by linearly projecting ε_t on ξ_t , we can decompose the static component as $\varepsilon_t = \rho\xi_t + \nu_t$, where $\rho = A_{\perp}A'_{\perp}\Sigma_u A(A'\Sigma_u A)^{-1}$, and then rewrite Equation (10) as

$$Y_t = \underbrace{C(L)\xi_t}_{\chi_t} + \nu_t, \quad (11)$$

where $C_0 = A + \rho$ and $C_j = A\gamma_j$ for $j > 0$.

Representation (11) highlights that the system dynamics are entirely generated by errors ξ_t . Hence, we label χ_t as the common component of Y_t and ν_t as the ignorable errors, as we assume the latter are not endowed with a structural interpretation. Since the errors ξ_t and ν_t are uncorrelated at any lead and lag, it is then possible to recover the structural shocks solely from the reduced form errors ξ_t of the common component χ_t using any of the procedures that are commonly employed in structural VAR analysis.

For instance, one may obtain the structural shocks as $u_t = C^{-1}D\xi_t$ and the impulse response functions from $\Psi(L) = C(L)D^{-1}C$, where D is the matrix formed by the first r rows of C_0 and C is a lower triangular matrix such that $CC' = DA'\Sigma_u AD'$. Since the first r rows of $\Psi(0)$, being equal to C , form a lower triangular matrix, the usual interpretation of the structural shocks u_t applies as long as the s ($s \leq r$) variables of interest are placed and properly ordered in the first s elements of Y_t . Notice that such identification strategy is based on a unique rotation of the reduced form common shocks ξ_t , and hence it does not require to endow the dynamic component x_t with an economic interpretation.

Clearly, it is always possible to identify the structural shocks directly from the reduced form errors u_t of the large VAR. However, the advantage of the approach based on representation (11) is that it requires to identify r shocks only instead of n of them. Hence, in the structural DRVAR analysis based on (11) we have a number of structural shocks that is smaller than the number of variables, as it is typical in both structural factor models (see e.g. Forni *et al.*, 2009) and dynamic stochastic general equilibrium (DSGE) models (see e.g. Fernández-Villaverde *et al.*, 2016).

2.2 Statistical inference

In order to consistently estimate the matrix A , we start by relying on the approach suggested by Lam *et al.* (2011), and Lam and Yao (2012). This approach has recently been extended in various directions, such as cointegration (Zhang *et al.*, 2019), principal component analysis for stationary time series (Chang *et al.*, 2018), and multivariate volatilities modelling (Tao *et al.* 2011; Li *et al.*, 2016).

Let us denote the autocovariance matrix of series Y_t at lag j as $\Sigma_y(j) = E(Y_t Y_{t-j}')$. In view of Equation (3), we see that

$$A'_\perp \Sigma_y(j) = E(A'_\perp \varepsilon_t Y_{t-j}') = 0 \quad \forall j > 0.$$

Hence, the matrix A lies in the space generated by the eigenvectors associated with the r non-zero eigenvalues of the symmetric and semi-positive definite matrix

$$M = \sum_{j=1}^{p_0} \Sigma_y(j) \Sigma_y(j)',$$

where p_0 is a positive integer. Given the assumption that series Y_t follow a finite order VAR(p) model, one would ideally fix $p_0 = p$.

Let us indicate with \hat{V}_q the matrix formed by the eigenvectors associated with the q ($\leq n$) largest eigenvalues of the matrix

$$\hat{M} = \sum_{j=1}^{p_0} \hat{\Sigma}_y(j) \hat{\Sigma}_y(j)',$$

where $\hat{\Sigma}_y(j)$ denotes the sample autocovariance matrix of Y_t at lag j .

Under regularity conditions that are compatible with our assumptions, \hat{V}_r estimates A (up to an orthonormal transformation) with a rate equal to $n^{-\delta} T^{1/2}$ when r is fixed, $n, T \rightarrow \infty$, and $\bar{a}'_i \bar{a}_i \asymp n^{1-\delta}$ for $i = 1, \dots, r$, where $\bar{A} = [\bar{a}_1, \dots, \bar{a}_r]$ and $\delta \in [0, 1]$. Notice that δ can be interpreted as an inverse measure of the strength of the factors: when $\delta = 0$ the factors are strong since the common component is shared by most of the n time series, whereas the factors are weak when $\delta \in (0, 1]$ (see Theorem 1 of Lam *et al.*, 2011). Moreover, Lam and Yao (2012) proved that a consistent estimator of r is provided by

$$\hat{r} = \arg \min_{i=1, \dots, R} \left\{ \hat{\lambda}_{i+1} / \hat{\lambda}_i \right\}, \quad (12)$$

where R is a constant such that $r < R < n$ and $\hat{\lambda}_i$ is the i -th largest eigenvalue of matrix \hat{M} .⁵

Remark 4 *Lam et al. (2011) and Lam and Yao (2012) assume that $\mathbb{E}(\varepsilon_{t+k}x_t') = 0$ for $k \geq 0$, whereas in our framework the strict inequality $k > 0$ only holds. However, we can always transform the original decomposition (3) in such a way that the two components are contemporaneously uncorrelated and the static component is still a white noise. Indeed, using the decomposition of the identity matrix proposed by Centoni and Cubadda (2003)*

$$\underbrace{A(A'\Sigma_u^{-1}A)^{-1}A'\Sigma_u^{-1}}_{\check{A}'} + \underbrace{\Sigma_u A_{\perp}(A'_{\perp}\Sigma_u A_{\perp})^{-1}A'_{\perp}}_{\check{A}_{\perp}} = I_n$$

we can decompose series Y_t as

$$Y_t = A\check{x}_t + \check{\varepsilon}_t,$$

where $\check{x}_t = \check{A}'Y_t$, $\check{\varepsilon}_t = \check{A}_{\perp}A'_{\perp}u_t$. It follows that

$$\mathbb{E}(\check{\varepsilon}_t\check{x}_t') = \check{A}_{\perp}A'_{\perp}\Sigma_u\check{A} = 0$$

and

$$\mathbb{E}(\check{\varepsilon}_t|F_{t-1}) = \check{A}_{\perp}A'_{\perp}\mathbb{E}(u_t|F_{t-1}) = 0.$$

Hence, in our framework the assumption that the dynamic component and the white noise are contemporaneously uncorrelated turns out to be unnecessary for the estimation of A and r through the eigen-analysis of the matrix M .

Remark 5 *When the factors are strong, i.e. $\delta = 0$, and the cross-correlation between the dynamic and static component is not so large to distort the information on the autocorrelation of the former, a "blessing of dimensionality" phenomenon occurs since the estimating accuracy of \hat{V}_r has the standard \sqrt{T} rate independently from the dimension n . The intuition is that the strong factors exploit the information coming from most of, if not all, the n series, hence the curse of dimensionality is offset by the increase of the information on the dynamic component (for further details and comments see Section 3 of Lam et al., 2011).*

Notice that \hat{r} in (12) consistently estimates the rank of the matrix M when Assumption 1 only applies, whereas we need an estimator of r that is subject to Assumption 2 as well. Let us first consider the problem of estimating the parameters of model (8) assuming that r is known and having fixed A equal to \hat{V}_r . In order to accomplish this goal, it is convenient to rewrite model (8) in its matrix form

$$Y = Z\alpha A' + u, \tag{13}$$

where $Y = [y_{p+1}, \dots, y_T]'$, $u = [u_{p+1}, \dots, u_T]'$, $z_t = [x_t', \dots, x_{t-p+1}']'$, and $Z = [z_p, \dots, z_{T-1}]'$. Then apply the Vec operator to both the sides of Equation (13) and use the property $\text{Vec}(ABC) = (C' \otimes A)\text{Vec}(B)$ to get

$$\text{Vec}(Y) = (A \otimes Z)\text{Vec}(\alpha) + \text{Vec}(u), \tag{14}$$

from which it is easy to see that the ordinary least squares (OLS) estimator of $\text{Vec}(\alpha)$ in Equation (14) takes the following form:

$$\text{Vec}(\hat{\alpha}) = [A' \otimes (Z'Z)^{-1}Z']\text{Vec}(Y). \tag{15}$$

⁵Li et al. (2017) proposed an improved estimator of r that is consistent even when not all the factors have the same strength. However, they take the assumption of independence between x_t and ε_t , which clearly does not hold here.

The main theoretical justification for considering the estimator (15) is that it is equivalent to applying OLS on (7),⁶ which turns out to be the quasi maximum likelihood (QML) estimator of parameters α in the small-scale VAR model of the factor x_t under the assumption that A is known.

An alternative estimator of parameters α can be obtained by applying the generalized least squares (GLS) on Equation (14). In particular, pre-multiply both the sides of Equation (14) by $\Sigma_u^{-1/2} \otimes I_{T-p}$ to get

$$(\Sigma_u^{-1/2} \otimes I_{T-p})\text{Vec}(Y) = \left(\Sigma_u^{-1/2} A \otimes Z \right) \text{Vec}(\alpha) + (\Sigma_u^{-1/2} \otimes I_{T-p})\text{Vec}(u). \quad (16)$$

Tedious but simple algebra reveals that the OLS estimator of $\text{Vec}(\alpha)$ in Equation (16) takes the following form:

$$\text{Vec}(\hat{\alpha}) = \left[(A' \Sigma_u^{-1} A)^{-1} A' \Sigma_u^{-1} \otimes (Z' Z)^{-1} Z' \right] \text{Vec}(Y). \quad (17)$$

In view of Equation (16), it is easy to see that the GLS estimator (17) is the QML estimator of parameters α in model (13) under the assumption that A and Σ_u are known.

The relation, in terms of efficiency, between the estimators (15) and (17) is provided in the following theorem.

Theorem 1 *Assuming that A and Σ_u are known, estimator (17) of $\text{Vec}(\alpha)$ has a mean square error matrix, conditionally on Z , that is not larger than the one of estimator (15). The two estimators have the same mean square error matrix when $A'u_t$ and $A'_\perp u_t$ are not correlated.*

Proof. See the appendix. ■

In order to derive a feasible GLS (FGLS) estimator, we suggest the following switching algorithm, which has the property to increase the Gaussian likelihood conditional to A in each step.

1. In view of Equation (8) and given (initial) estimates of α , maximize the conditional Gaussian likelihood $\mathcal{L}(\Sigma_u | \alpha, A)$ by estimating Σ_u with

$$(T-p)^{-1} (Y' - A\alpha'Z')(Y - Z\alpha A').$$

2. Given the previously obtained estimate of Σ_u , maximize $\mathcal{L}(\alpha | \Sigma_u, A)$ by estimating elements of α with (17).
3. Repeat steps 1 and 2 till numerical convergence occurs.⁷

In order to speed up the numerical convergence of that algorithm, it is important to choose the initial values for the coefficient matrix α correctly. An obvious choice is resorting to $\hat{\alpha}$, which provides a consistent estimate of α as T increases.

A practical problem that arises when the sample size T and the dimension n are of similar magnitude is that the estimate of matrix Σ_u is singular or nearly singular. We propose to solve this problem by ignoring the error cross-correlations in the estimation method. In particular, we suggest to use a diagonal matrix Δ_u with the same diagonal as Σ_u in place of Σ_u itself in the FGLS procedure. This solution has two main motivations. First, it makes the objective function of the switching algorithm to become $\text{trace}(\ln(\Delta_u))$,

⁶This result is obtained by post-multiplying with A both sides of Equation (13) and then applying the Vec operator to get

$$\text{Vec}(YA) = (I_r \otimes Z)\text{Vec}(\alpha) + \text{Vec}(\epsilon A)$$

It is easy to see that the OLS estimator of $\text{Vec}(\alpha)$ in the model above is the same as (15).

⁷A general proof of the convergence of this family of iterative procedures is given by Oberhofer and Kmenta (1974).

which is a common approximation of $\ln(\det(\Sigma_u))$ in high-dimensional settings, see Hu *et al.* (2017) and the references therein. Second, it is reasonable to presume that the fraction of unanticipated co-movements among variables is small when the conditioning information set is large.

Finally, in order to identify the dimension the dynamic component r , we suggest the following strategy. For $q = 1, \dots, R$ estimate either by OLS or FGLS the models

$$Y_t = \sum_{j=1}^p \hat{V}_q \alpha_{j,q} \hat{V}_q' Y_{t-j} + u_t(q),$$

where $\alpha_{j,q}$ is a $q \times q$ matrix for $j = 1, \dots, p$, and estimate r as the index \hat{r} that minimizes an information criterion such as

$$\text{IC}(q) = n^{-1} \ln \prod_{i=1}^n \hat{\sigma}_i^2(q) + \frac{c_T k}{Tn} \quad (18)$$

where $\hat{\sigma}_i^2(q) = (T-p)^{-1} \sum_{t=p+1}^T u_{t,i}^2(q)$, $u_t(q) = [u_{t,1}(q), \dots, u_{t,n}(q)]'$, $k = nq + (p-1)q^2$, c_T is a penalty term such that $c_T = 2$ for the Akaike information criterion (AIC), $c_T = 2 \ln(\ln(T))$ for the Hannan-Quinn information criterion (HQIC), and $c_T = \ln(T)$ for the Bayes information criterion (BIC). Notice that the measure of fit to be used is $\text{trace}(\ln(\Delta_u))$, given the assumption that $\Sigma_u = \Delta_u$, and the overall number of parameters is $k = nq + (p-1)q^2$, given that the number of free parameters in a base of the space spanned by A is equal to $nq - q^2$ and each of the p $\alpha_{j,q}$ matrices has q^2 coefficients.

The asymptotic behavior of the information criteria (18) is given in the following proposition.

Proposition 1 *Under conditions such that OLS and FGLS estimate the DRVAR parameters (up to an orthonormal transformation) with the standard \sqrt{T} rate as $n, T \rightarrow \infty$, and assuming that $\gamma = \lim_{n \rightarrow \infty} \left(\prod_{i=1}^n \sigma_i^2 \right)^{1/n}$ exists, where $\text{diag}[\sigma_1^2, \dots, \sigma_n^2] = \Delta_u$, the BIC and HQIC provide weakly consistent estimators for the number of dynamic components r but not for the overall number of the DRVAR parameters k .*

Proof. See the appendix. ■

3 Monte Carlo analysis

3.1 The data generating process

In this section we perform a Monte Carlo study to evaluate the finite sample performances of the OLS and FGLS estimators of model (5) parameters having estimated the matrix A according to Lam *et al.* (2011) in both cases. We consider the following n -dimensional stationary VAR(2) process

$$Y_t = \underbrace{\bar{A} \text{diag}(\delta_1) \bar{A}^+}_{\Phi_1} Y_{t-1} + \underbrace{\bar{A} \text{diag}(\delta_2) \bar{A}^+}_{\Phi_2} Y_{t-2} + u_t, \quad (19)$$

where \bar{A} is a $n \times r$ matrix such that its columns are generated by r i.i.d. $N_n(0, I_n)$, $\bar{A}^+ = (\bar{A}'\bar{A})^{-1}\bar{A}'$ is the Moore–Penrose pseudo inverse of the matrix \bar{A} , $\delta_1 = 2\text{diag}(m) \cos(\omega)$, m is a r -vector drawn from a $U_r[0.3, 0.9]$, ω is a r -vector drawn from a $U_r[0, \pi]$, $\delta_2 = -m^2$, and u_t are i.i.d. $N_n(0, \Sigma_u)$.⁸ Notice that Equation (19) is equivalent to

$$Y_t = A\alpha_1 A' Y_{t-1} + A\alpha_2 A' Y_{t-2} + u_t,$$

⁸Notice that the squared Euclidean norms of the columns of \bar{A} are proportional to n on average over the replications. However, the randomness of the matrix \bar{A} ensures for each replication some variability of the degree of strength over the r factors.

where $A = \bar{A}S^{-1}$, $S = (\bar{A}'\bar{A})^{1/2}$, and $\alpha_i = S\text{diag}(\delta_i)S^{-1}$ for $i = 1, 2$ and consequently imposes Assumptions 1 and 2 of our specification.

In order to simulate series Y_t , we first generate the diagonal VAR(2) process

$$\bar{x}_t = \text{diag}(\delta_1)\bar{x}_{t-1} + \text{diag}(\delta_2)\bar{x}_{t-2} + \bar{A}^+u_t$$

and then the static component $\varepsilon_t = \bar{A}_\perp\bar{A}_\perp^+u_t$ with $\eta_t = [(\bar{A}^+u_t)', (\bar{A}_\perp^+u_t)']'$ that are i.i.d. $N_n(0, \Sigma_\eta)$. We can finally obtain

$$Y_t = \bar{A}\bar{x}_t + \varepsilon_t = Ax_t + \varepsilon_t, \quad (20)$$

where $x_t = S\bar{x}_t$ is the dynamic component.

An important role in the data generating process is played by the covariance matrix Σ_η , which has the following Toeplitz structure

$$\Sigma_\eta = \begin{bmatrix} 1 & \tau & \tau^2 & \tau^3 & \dots & \tau^n \\ \tau & 1 & \tau & \tau^2 & \dots & \tau^{n-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \tau^n & \tau^{n-1} & \tau^{n-2} & \tau^{n-3} & \dots & 1 \end{bmatrix},$$

where τ is a scalar drawn from a $U[-0.5, 0.5]$. Notice that since Σ_η is not diagonal by implication the covariance matrix of the VAR errors Σ_u is not diagonal as well. This allows us to evaluate the performances of both the FGLS estimator and the information criteria when $\ln(\det(\Sigma_u))$ is not equal to $\text{trace}(\ln(\Delta_u))$.

3.2 Results

From (20) we generate systems of successively $n = 150, 300, 600, 1200$ variables. We consider $r = 3, 9$ and n dynamic components; $r = 3$ is indeed often assumed in financial applications (the Fama-French factors) whereas several studies find that there exist from around 8 up to 10 factors in large macroeconomic datasets. The case $r = n$ is considered to evaluate the performances of the various estimators of r when Assumptions 1 and 2 are not valid.⁹

The number of observations is successively $T = \frac{1}{2}n, n, 1.5n$. We consequently evaluate the performance of our approach when the number of variables is respectively less, equal or larger than the sample size. We simulate $T + 50$ observations and the first 50 points are used as a burn-in period, the remaining ones for estimations. Results are based on 1000 replications.

The proposed methods are evaluated by means of two statistics. We first compute the percentage with which the number of dynamic components r is either correctly identified ($\% \hat{r} = r$) when $r = 3, 9$ or hits the upper bound ($\% \hat{r} = R = 11$) when $r = n$ using both the estimator (12) proposed by Lam and Yao (2012), hereafter denoted as LY, and the usual information criteria under the assumption that Σ_u is a diagonal matrix. We also compute the average of \hat{r} over the replications as well as the frequencies with which those procedures underestimate the correct number of dynamic components ($\% \hat{r} < r$) when $r < n$. Second, when the DRVAR restrictions are valid, i.e. $r = 3, 9$, we compute the Frobenius distance between the estimates of $\Phi = [\Phi_1, \Phi_2]'$ and the true ones relative to the Frobenius norm of Φ (RFD) as a measure of estimation precision. We only document the results for the LY procedure and the OLS estimator since those with FGLS are rather similar to the OLS ones and are then omitted for the sake of space.¹⁰

⁹Notice that when $r = n$ the DGP(20) boils down to $Y_t = \bar{A}\bar{x}_t$, where \bar{A} is a $n \times n$ matrix such that its columns are generated by n i.i.d. $N_n(0, I_n)$ and \bar{x}_t is generated by a n -dimensional diagonal VAR(2) process.

¹⁰In particular, the percentages of correct identification that are obtained by each criterion are almost identical irrespective of

Table 1: Monte Carlo results, $r = 3$, OLS estimator

T/n		$n = 150$				$n = 300$			
		$\% \hat{r} = 3$	$\% \hat{r} < 3$	$\bar{\hat{r}}$	RFD	$\% \hat{r} = 3$	$\% \hat{r} < 3$	$\bar{\hat{r}}$	RFD
$T = \frac{n}{2}$	LY	40.8	57.1	2.100	-	53.7	43.0	2.389	-
	BIC	67.2	32.0	2.605	39.46	78.6	21.3	2.748	27.55
	HQIC	72.6	12.1	3.094	74.88	86.6	7.5	2.993	42.44
	AIC	29.1	3.2	5.983	274.27	42.5	1.7	4.801	192.21
$T = n$	LY	52.8	45.1	2.311	-	62.3	35.2	2.493	-
	BIC	80.3	19.5	2.784	26.94	90.0	9.90	2.898	18.56
	HQIC	86.2	7.6	3.002	38.11	92.7	4.1	2.992	23.72
	AIC	45.1	1.4	4.633	144.22	51.1	0.7	4.444	122.71
$T = 1.5n$	LY	55.8	43.1	2.328	-	64.3	32.7	2.555	-
	BIC	83.6	16.2	2.824	22.15	91.8	8.2	2.917	14.916
	HQIC	91.6	4.4	3.005	27.93	95.0	2.4	3.003	19.140
	AIC	53.5	0.7	4.245	105.08	52.8	0.1	4.217	99.841
		$n = 600$				$n = 1200$			
	IC	$\% \hat{r} = 3$	$\% \hat{r} < 3$	$\bar{\hat{r}}$	RFD	$\% \hat{r} = 3$	$\% \hat{r} < 3$	$\bar{\hat{r}}$	RFD
$T = \frac{n}{2}$	LY	59.9	35.1	2.560	-	65.7	27.8	2.734	-
	BIC	89.6	10.4	2.882	18.39	94.0	6.00	2.936	12.89
	HQIC	92.5	2.9	3.017	29.07	95.5	2.2	3.003	17.84
	AIC	51.5	0.6	4.286	147.14	52.8	0.1	4.171	137.20
$T = n$	LY	67.7	27.9	2.683	-	73.2	21.5	2.815	-
	BIC	94.0	5.8	2.944	13.06	97.6	2.4	2.976	8.85
	HQIC	96.8	1.3	3.009	16.31	97.4	0.8	3.010	12.36
	AIC	52.2	0.1	4.167	110.85	52.0	0.0	4.176	112.70
$T = 1.5n$	LY	73.5	22.6	2.755	-	77.5	18.2	2.819	-
	BIC	96.5	3.5	2.964	10.54	99.1	0.9	2.991	7.27
	HQIC	96.4	0.8	3.021	15.08	97.7	0.0	3.025	11.72
	AIC	50.8	0.0	4.162	98.16	52.3	0.0	4.127	94.88

Notes: Percentages with which each method correctly estimates or underestimates the true r , the average of estimates of r over 1000 replications, and the Frobenius distance between Φ and its estimates relative to the Frobenius norm of Φ .

Table 2: Monte Carlo results, $r = 9$, OLS estimator

T/n		$n = 150$				$n = 300$				
		$\% \hat{r} = 9$	$\% \hat{r} < 9$	$\bar{\hat{r}}$	RFD	$\% \hat{r} = 9$	$\% \hat{r} < 9$	$\bar{\hat{r}}$	RFD	
$T = \frac{n}{2}$	LY	22.7	77.3	3.639	-	39.2	60.8	4.753	-	
	BIC	44.7	54.5	7.781	59.54	50.0	50.0	8.135	38.57	
	HQIC	63.1	9.8	9.262	109.20	85.7	9.1	8.946	48.29	
	AIC	18.7	0.3	10.405	202.09	35.2	0.1	10.059	153.15	
$T = n$	LY	34.5	65.5	4.443	-	47.1	52.9	5.389	-	
	BIC	47.9	52.1	8.108	39.41	61.4	38.6	8.459	27.38	
	HQ	83.0	11.6	8.928	46.82	91.4	6.3	8.958	30.55	
	AIC	41.3	0.6	9.911	115.60	45.5	0	9.836	100.99	
$T = 1.5n$	LY	42.7	57.3	5.075	-	56.8	43.2	6.019	-	
	BIC	59.8	40.2	8.386	32.17	74.3	25.7	8.681	22.04	
	HQIC	90.2	7.6	8.940	34.47	94.8	3.9	8.974	23.45	
	AIC	48.9	0.2	9.765	88.17	51.4	0.0	9.729	78.47	
		$n = 600$				$n = 1200$				
		IC	$\% \hat{r} = 9$	$\% \hat{r} < 9$	$\bar{\hat{r}}$	RFD	$\% \hat{r} = 9$	$\% \hat{r} < 9$	$\bar{\hat{r}}$	RFD
$T = \frac{n}{2}$	LY		51.1	48.9	5.534	-	59.5	40.5	6.162	-
	BIC		61.1	38.8	8.442	27.03	73.3	26.7	8.676	18.96
	HQIC		91.7	6.2	8.953	30.69	96.1	3.4	8.970	19.98
	AIC		41.6	0.2	9.893	131.91	46.9	0.0	9.800	119.15
$T = n$	LY		64.4	35.6	6.540	-	74.2	25.8	7.164	-
	BIC		76.2	23.8	8.719	18.83	88.9	11.1	8.873	13.16
	HQIC		96.8	2.4	8.984	19.85	99.5	0.3	8.998	13.47
	AIC		48.8	0.0	9.760	90.40	45.6	0.0	9.818	93.59
$T = 1.5n$	LY		64.9	35.1	6.559	-	75.7	24.3	7.285	-
	BIC		83.1	16.9	8.809	15.25	93.1	6.9	8.927	10.58
	HQIC		97.9	1.7	8.986	15.54	99.5	0.2	9.001	10.97
	AIC		51.1	0.0	9.746	74.52	48.0	0.0	9.800	76.15

See the notes of Table 1.

Table 3: Monte Carlo results, $r = n$, OLS estimator

T/n		$n = 150$		$n = 300$		$n = 600$		$n = 1200$	
		$\% \hat{r} = R$	\hat{r}	$\% \hat{r} = R$	\hat{r}	$\% \hat{r} = R$	\hat{r}	$\% \hat{r} = R$	\hat{r}
$T = \frac{n}{2}$	LY	0.0	2.022	0.10	2.028	0.10	2.156	0.30	2.316
	BIC	0.0	5.178	15.30	8.116	84.40	10.718	100.0	11.000
	HQIC	46.4	9.638	94.60	10.914	100.0	11.000	100.0	11.000
	AIC	96.7	10.962	99.80	10.998	100.0	11.000	100.0	11.000
$T = n$	LY	0.30	2.032	0.10	2.252	0.40	2.281	0.50	2.443
	BIC	13.00	7.939	85.40	10.744	100.0	11.000	100.0	11.000
	HQIC	85.4	10.756	99.80	10.998	100.0	11.000	100.0	11.000
	AIC	99.7	10.997	100.0	11.000	100.0	11.000	100.0	11.000
$T = 1.5n$	LY	0.10	2.143	0.00	2.204	0.20	2.428	0.70	2.486
	BIC	51.70	9.885	98.70	10.987	100.0	11.000	100.0	11.000
	HQIC	97.80	10.973	100.0	11.000	100.0	11.000	100.0	11.000
	AIC	100.0	11.000	100.0	11.000	100.0	11.000	100.0	11.000

See the notes of Table 1.

We first examine the results for $r = 3, 9$, which are respectively reported in Tables 1 and 2. As expected, we notice that all methods perform better as the dimension of the system n increases and, conditional to a given n , as the sample size T gets larger. For what regards the estimation of the true number of dynamic components, we see that the information criteria outperform the LY procedure by a clear margin. This finding is hardly surprising given the parametric nature of the information criteria. In particular, HQIC identifies the correct model better than the competitors but in 3 cases where BIC performs best. In contrast, LY [AIC] systematically underestimates [overestimates] the true r . With respect to the RFD, we observe that the models identified by the BIC [AIC] provide estimates of Φ that are more [less] accurate than those obtained by the other criteria over all the settings. Such outcomes are likely due the fact that the BIC is downward biased, thus implicitly shrinking to zero the dynamic components that are only mildly autocorrelated, whereas the AIC systematically overfits the model, thus inflating the estimation error. This finding discourages the use of the AIC in empirical applications. Overall, BIC and HQIC tend to perform similarly in both identification and estimation precision as both n and T get large.

Table 3 reports the results when the DRVAR restrictions are not valid, i.e. for $r = n$. We see that the LY procedure, which is designed to estimate r when Assumption 1 is valid, spuriously suggests the presence of about two dynamic components on average irrespective of the sample size and the system dimension. In contrast, the usual information criteria, especially HQIC and AIC, provide estimates of r that get close or hit the upper bound starting from $n, T = 300, 300$ up to larger values of both n and T . These findings indicate that, when there exist no common dynamic components in the data, the information criteria (18) correctly provide estimates of r that tend to become larger as n, T increase.

Finally, as suggested by a referee we build an alternative Monte Carlo design that is based on the features of the data that we use in Section 4. Our favorite DRVAR empirical specification is a model with $r = 8$ and

the estimation method, whereas FGLS generally exhibits slightly lower RFDs than OLS, although the differences are significant in about one third of the cases. Results are available upon request.

Table 4: Monte Carlo results, data-based DGP, OLS estimator

n/T		$T = 242$				$T = 484$			
		$\% \hat{r} = 8$	$\% \hat{r} < 8$	\hat{r}	RFD	$\% \hat{r} = 8$	$\% \hat{r} < 8$	\hat{r}	RFD
$n = 211$	LY	0.0	100.0	1.735	-	1.8	98.2	1.816	
	BIC	8.8	91.2	6.464	78.93	55.0	45.0	7.548	55.97
	HQIC	62.9	34.7	7.624	75.05	96.6	1.8	7.998	53.13
	AIC	52.8	3.1	8.709	77.00	71.8	0.0	8.448	54.25

Notes: The DGP has the same parameters as the empirical model in Section 4 and Gaussian errors. See the notes of Table 1 for the notation.

$p = 2$ for a dataset of 211 aggregate US time series and 242 observations. Obviously, n cannot be increased beyond 211 in this framework but we can evaluate the effects of doubling the sample size T on the proposed methods. Hence, data are generated by a DRVAR having Gaussian errors and the same parameters as the model that we estimate in Section 4. The results, reported in 4, confirm most of the main findings of the previous experiments, with the exceptions that AIC outperforms BIC and that LY behaves even more poorly than it does with the artificial DGP.¹¹

Remarkably, the general outcome that (consistent) information criteria are particularly useful in selection of VAR models is fully in line with previous contributions in the literature (see e.g. Cavaliere et al., 2015, 2018; Kapetanios, 2004; Nielsen, 2006). Regarding the poor performances of the LY procedure, we remark that, differently from our DGPs, the dynamic component and the white noise are generated independently in the Monte Carlo study by Lam in Yao (2012). As noted by Lam *et al.* (2011), the cross correlation between the dynamic and the static component is beneficial to the estimation of A , resulting even in a faster convergence rate as $n^{-\delta/2}T^{1/2}$ when such cross correlation is strong, whereas it creates difficulties in the estimation of r by the LY procedure, see the asymptotic analysis in Lam in Yao (2012).

4 Empirical application

This section illustrates the feasibility and the practical value of our dimension reduction approach to VAR modelling. We first search for the presence of co-movements among 211 US quarterly economic and financial time series. Based on our methodology, we then propose a novel approach to identify the shock that is responsible for most of the variability of the common components at the business cycle frequencies.

4.1 Co-movements in quarterly US time series

We start by investigating a dimension reduction for a high-dimensional VAR of the US economy. The data are obtained from the Federal Reserve Economic Quarterly Database (FRED-QD henceforth), to which we added the total factor productivity time series corrected for utilization produced by Fernald (2012).¹² FRED-QD is regularly updated and different releases of all the series are available online. A detailed description

¹¹Again, the RFDs of FGLS are about 1% lower than those of OLS but the differences are significant in all cases with the data-based DGP.

¹²This variable is relevant for the structural analysis that we conduct in the next subsection.

of the variables and the proposed transformations used to achieve stationarity of each series is provided by McCracken and Ng (2020).

After some necessary cleaning of the dataset and various stationarity transformations of the series, we have at disposal $n = 211$ variables with $T = 242$ quarterly observations from 1959Q3 up to 2019Q4.¹³ For comparison convenience, we use the transformations proposed in FRED-QD to make the time series stationary, although the results of unit root tests might suggest alternative transformations in some cases. Moreover, series are demeaned and standardized to have a unit variance after having corrected them for outliers.¹⁴

We start the analysis by fixing $p_0 = 5$, a rather typical lag length of a VAR model for quarterly data, and $R = 14$ as the upper bound of the dimension of the dynamic component.¹⁵ The LY procedure detects $r = 2$ dynamic components. This finding is rather dubious given the huge heterogeneity in the series we work with. In order to determine the largest VAR order, we use the traditional information criteria to estimate the lag length in a VAR model for series $\hat{V}'_R Y_t$.¹⁶ We get $p = 1$ according to the BIC, $p = 2$ according to the HQIC, and $p = 4$ according to the AIC. Consequently, we consider successively $p = 1, \dots, 4$ lags when estimating r through the information criteria (18) using OLS and FGLS in estimation. As in the Monte Carlo study, the two estimation methods provide identical estimates of r : BIC [AIC] systematically indicates $r = 7$ [$r = 14$], whereas HQIC indicates either $r = 8$ or $r = 12$ according to the VAR lag length. After a careful comparison of the various specifications, we opt for $r = 8$ and $p = 2$, with the latter being the indication coming from the HQIC when it is used to determine the lag length having fixed $r = 8$.¹⁷ We remark that the presence of eight common components is a rather typical finding in the empirical literature on factor models using similar data as ours.

Next, we compute two statistics in order to evaluate how the model fits to the data. First, we consider the coefficients of determination of each element of Y_t as obtained by model (8). Second, we compute the squared correlation coefficients between each element of Y_t and its counterpart in the common component χ_t of representation (11). We denote the former statistic as $R_{Y,Z}^2$ and the latter as $R_{Y,\Xi}^2$. It is easy to see that $R_{Y,\Xi}^2 \geq R_{Y,Z}^2$.

Whereas $R_{Y,Z}^2$ has the usual interpretation in terms of measure of the degree of predictability, $R_{Y,\Xi}^2$ indicates the fraction of the variability of each element of Y_t that is explained by a linear projection on the present and past values of the dynamic errors ξ_t . Hence, $R_{Y,\Xi}^2$ measures the importance of the common component χ_t in the variability of each series.¹⁸

Based on the FGLS estimates of the coefficients of the DRVAR model, we report in Table 5 the averages as well as the quartiles of the empirical distributions of both $R_{Y,Z}^2$ and $R_{Y,\Xi}^2$.

Moreover, in Table 6 we report the estimates of both $R_{Y,Z}^2$ and $R_{Y,\Xi}^2$ for nine macroeconomic variables that we are going to analyze in the subsequent subsection: Output (GDP), Consumption (Con), Investment (Inv), Unemployment Rate, (UR), Worked Hours (Hours), Inflation Rate (Inf), Interest Rate (IR), Labor Productivity (LP), and Total Factor Productivity (TFP). The exact denominations of the variables along

¹³We did not include the variables that were not observed from 1959Q1 up to 2019Q4.

¹⁴In particular, we removed 20 outliers from 16 series. Our dataset is available upon request.

¹⁵The results that will be reported later are robust to alternative choices as $p_0 = 3, \dots, 9$ and $R = 12, \dots, 15$.

¹⁶Notice that, if series Y_t are generated by a DVAR model, then the linear combinations $V'_R Y_t$ follow a VAR model of the same order as the dynamic component x_t .

¹⁷Choosing either $r = 12$ or $p = 1$ leads to empirical results of the subsequent analysis that are qualitatively very similar to the ones that will be documented later.

¹⁸Notice that $R_{Y,\Xi}^2$ is actually computed as the sample analogous of the squared correlation coefficient between each element of Y_t and the corresponding element in $Y_t - \nu_t$.

Table 5: Average and quartiles of the measures of fit

	Mean	Q_1	Q_2	Q_3
$R_{Y,Z}^2$	0.30	0.12	0.25	0.47
$R_{Y,\Xi}^2$	0.53	0.34	0.56	0.73

with their stationarity transformations are reported in the appendix.

Table 6: Measures of fit for 9 key aggregate variables

	GDP	Con	Inv	UR	Hours	Inf	IR	LP	TFP
$R_{Y,Z}^2$	0.40	0.38	0.40	0.64	0.61	0.22	0.28	0.20	0.10
$R_{Y,\Xi}^2$	0.83	0.71	0.73	0.89	0.85	0.90	0.68	0.66	0.37

We see that, as expected, the estimates of $R_{Y,\Xi}^2$ are considerably larger than those of $R_{Y,Z}^2$ over all the series and the nine key variables as well. Remarkably, the role of the common component of TFP is smaller than the one of the other key variables. This finding may reflect the partial exogenous nature of TFP, as well as the possible presence of large estimation errors in a variable that it is not directly observable.

4.2 Comparison with a FAVAR model

In light of Remark 4, it is of interest to check whether a FAVAR or an (unrestricted) DRVAR fits better to the data. The first step of such comparison requires to estimate a FAVAR model on our dataset. We use the nine key variables as the observed factors and the remaining variables to construct the unobserved factors, which are estimated by the principal components of the remaining 202 variables.

In order to fix the number of unobserved factors, rather than relying on criteria that take into account the internal variability of the predictors only, we follow Pesaran *et al.* (2011) and use the traditional information criteria in a predictive model where the target variables are the nine key series and the predictors are two lags of both the targets and the estimates of the unobserved factors. The BIC and HQIC respectively suggest the presence of 1 and 3 unobserved factors, whereas the AIC hits the upper bound. As in the case of the DRVAR, we follow the indication coming from the HQIC.

In light of Proposition 1, it is not obvious how to rank the two empirical models at the system level. A possible solution is contrasting the alternative specifications of the partial model of the key series only (i.e., the equations of the large VAR corresponding to the nine key variables). Given the different number of parameters in the DRVAR and FAVAR specifications of the considered partial model, a simple comparison of the respective log-likelihoods would be misleading. Hence, Table 7 reports the results of the application of traditional information criteria to the competing specifications of the partial model for the key series, where the estimated factors are treated as observable in the computation of the penalty terms.¹⁹

We see that BIC clearly supports the DRVAR specification, whereas AIC [HQIC] favors the FAVAR [DRVAR]. Hence, notwithstanding the specification of the FAVAR is tailored for the key variables, the two

¹⁹Since in both models the unobservable factors are estimated by the eigenvectors of large covariance matrices, the sample variability of the two estimators is expected to be similar.

Table 7: Information criteria of the DRVAR and FAVAR partial models of the key series

	BIC	HQIC	AIC
DRVAR	-7.52	-8.62	-9.37
FAVAR	-6.62	-8.48	-9.73

empirical models fit similarly to the key series.

4.3 Identification of the shock driving the business cycle

A relevant issue in macroeconomics is the identifications of the shocks that drive the macroeconomic fluctuations. The textbook approach consists in identifying a given structural shock according to the guidance of economic theory (e.g. a productivity shock or a monetary shock) and to evaluate its impact over the key macroeconomic variables at various time horizons by means of the impulse response function and the forecast error variance decomposition.

Recently, this strategy has been subject of some criticism on the ground that empirical findings are strictly conditional on the validity of the underlying economic assumption. For instance, it may be too restrictive to assume that technology is the only shock that can permanently affect productivity. Alternatively, several authors have resorted to identification schemes that are based on the max-share identification strategy, as originally proposed by Uhlig (2003). The max-share methodology is a kind of reverse engineering approach, through which a shocks is identified as the main driver of a macroeconomic variable at a given time horizon. Contributions along this line of research include Barsky and Sims (2011) and Francis *et al.* (2014). Lately, Angeletos *et al.* (2020) proposed a max-share approach to identify the main driver of the business cycle. In particular, they identify the main business cycle shock as the shock that maximizes the volatility at the business cycle frequency band of a target variable. Using a Bayesian VAR model for ten macroeconomic variables, they show that alternatively targeting unemployment, output, hours worked, consumption and investment, their approach provides very similar impulse responses functions for ten key macroeconomic variables.

The approach that we adopt here is similar as the one by Angeletos *et al.* (2020) but with some relevant differences. First, we rely on a much richer information set coming from the large dimensional VAR that we have previously estimated. Second, we recognize that the business cycle is inherently a multivariate phenomenon and we aim at disentangling a unique driver of the business cycle for the whole economy rather than targeting a specific variable. Third, we search for the shock of the direction that maximizes the variability at the business cycle frequencies of the common component χ_t in decomposition (11). In this way, we are able to filter out the effect of the ignorable errors, which cannot generate cyclical fluctuations by construction but still contaminate the observed variables Y_t . Notice that the last goal could not be pursued simply by estimating a large VAR with some shrinkage method.

Formally, in view of Equation (11), the spectral density matrix of the common component χ_t is

$$F_{\chi}(\varpi) = (2\pi)^{-1}C(z^{-1})A'\Sigma_uAC(z)', \quad (21)$$

where $z = \exp(-i\varpi)$ and $\varpi \in [0, 2\pi)$.

Given that $\text{Re } F_{\chi}(\varpi)$ is proportional to the variance matrix of the ϖ -frequency component in the spectral

representation of χ_t (see, e.g., Subsections 4.6 and 7.1 in Brillinger, 2001), the matrix

$$\Theta(\varpi_0, \varpi_1) = \int_{\varpi_0}^{\varpi_1} \text{Re } F_{\varkappa}(\varpi) d\varpi$$

measures the (co-)volatility of the common component χ_t at the frequency band $[\varpi_0, \varpi_1]$, where $0 < \varpi_0 < \varpi_1 < \pi$.

Let Q be the matrix formed by the eigenvectors that are associated with the first r non-increasing eigenvalues of the matrix $\Theta(\varpi_0, \varpi_1)$, then the linear combinations $Q'\chi_t$ represent the (static) principal components of χ_t at the frequency band $[\varpi_0, \varpi_1]$. The Wold representation and the structural vector moving average representation of $Q'\chi_t$ are respectively given by

$$Q'\chi_t = Q(L)D\xi_t = Q'\Psi(L)u_t,$$

where $D = Q'C_0$, $Q(L) = Q'C(L)D^{-1}$, $\Psi(L) = C(L)D^{-1}C$, $u_t = C^{-1}D\xi_t$, and C is a lower triangular matrix such that $CC' = DA'\Sigma_u AD'$.

When $[\varpi_0, \varpi_1]$ is the typical business cycle frequency band, i.e. $[2\pi/32, 2\pi/6]$ for quarterly data, we label the first element in u_t as the Main Business Cycle Common Shock (MBCCS). In words, the MBCCS is the (standardized) shock of the direction that maximizes the contemporaneous variability of the common component χ_t at frequencies corresponding to periods between 6 and 32 quarters.²⁰ It is easy to see that the MBCCS is

$$u_{1t} = c_{11}^{-1}D'_{1\bullet}\xi_t$$

where c_{11} is the entry at the first row and first column of C and $D'_{1\bullet}$ is the first row of D , whereas, in light of decomposition (11), the associated impulse response function (IRF) for series Y_t is

$$\Psi_{\bullet 1}(L) = C(L)D^{-1}C_{\bullet 1},$$

where $C_{\bullet 1}$ is the first column of C .

Given the frequency domain nature of our identification scheme, we evaluate the effects of the MBCCS on the variable of interest over frequencies rather than over time horizons (Centoni and Cubadda, 2003; Angeletos *et al.*, 2020). In particular, we look at the contribution of the MBCCS to the variability of the i -th series at the business cycle frequency band

$$\frac{\int_{\pi/16}^{\pi/3} e_i'\Psi_{\bullet 1}(z^{-1})\Psi_{\bullet 1}(z)'e_i d\varpi}{2\pi \int_{\pi/16}^{\pi/3} e_i'F_Y(\varpi)e_i d\varpi},$$

and at the zero frequency

$$\frac{e_i'\Psi_{\bullet 1}(1)\Psi_{\bullet 1}(1)'e_i}{2\pi e_i'F_Y(0)e_i},$$

where e_i is an n -vector with unity as its i -th element and zeros elsewhere, and the spectral density matrix of series Y_t is

$$F_Y(\varpi) = F_{\varkappa}(\varpi) + (2\pi)^{-1}E(\nu_t\nu_t')$$

Regarding the computational aspects, we truncate the order of the estimated polynomial matrix $C(L)$ by setting $\hat{C}_j = 0$ for $j > 199$ and we approximate integrals over the business cycle frequency band with sums over 100 evenly spaced frequencies from $\pi/16$ up to $\pi/3$.

²⁰Notice that, since we are considering the real part only of the spectral density matrix $F_{\varkappa}(\varpi)$, we are implicitly focusing on the waves of χ_t at the frequency band $[\pi/16, \pi/3]$ that move in phase.



Figure 1: Estimate of the main business cycle common component

We start the empirical analysis by computing the sample eigenvalues of matrix $\Theta(\pi/16, \pi/3)$. The largest eigenvalue, namely the one that is associated with the MBCCS, accounts for about 54.8% of the variability of the common component at the business cycle frequency band. Figure (1) shows the estimate of the linear combination of the common component that is associated with the MBCCS, i.e. $Q'_{1\bullet}\chi_t$ where $Q'_{1\bullet}$ is the first row of Q , which we label as the Main Business Cycle Common Component (MBCCC). Shaded areas in the figure represent NBER-defined recessions. We see that the MBCCC accurately reproduces the main features of the US aggregate cycle. Moreover, its estimated spectrum exhibits a unique large peak corresponding to fluctuations with a period of about five years.

We report in Table 8 the estimates of the contributions of the MBCCS to the variability of the 9 key series at periods 6-32 and ∞ . This information helps to assess the role of the MBCCS both at the business cycle frequencies and in the long-run.

Table 8: Contributions of the MBCCS to the variability of the 9 key series at frequencies $[\pi/16, \pi/3]$ and 0

Period	GDP	Con	Inv	UR	Hours	Inf	IR	LP	TFP
6 – 32	40.2	15.3	47.9	44.2	43.7	8.7	28.9	21.0	5.4
∞	33.3	16.8	35.8	39.4	37.7	7.2	19.4	14.4	2.4

Finally, Figure (2) reports the estimates of the cumulated IRF to the MBCCS of the 9 variables already considered in the previous subsection. The IRF are cumulated to provide readers with the dynamic effects of the MBCCS on the levels of variables, thus facilitating the comparison with results of previous studies.

We see in Figure (2) that the MBCCS triggers a procyclical effect on GDP and Inv peaking with one

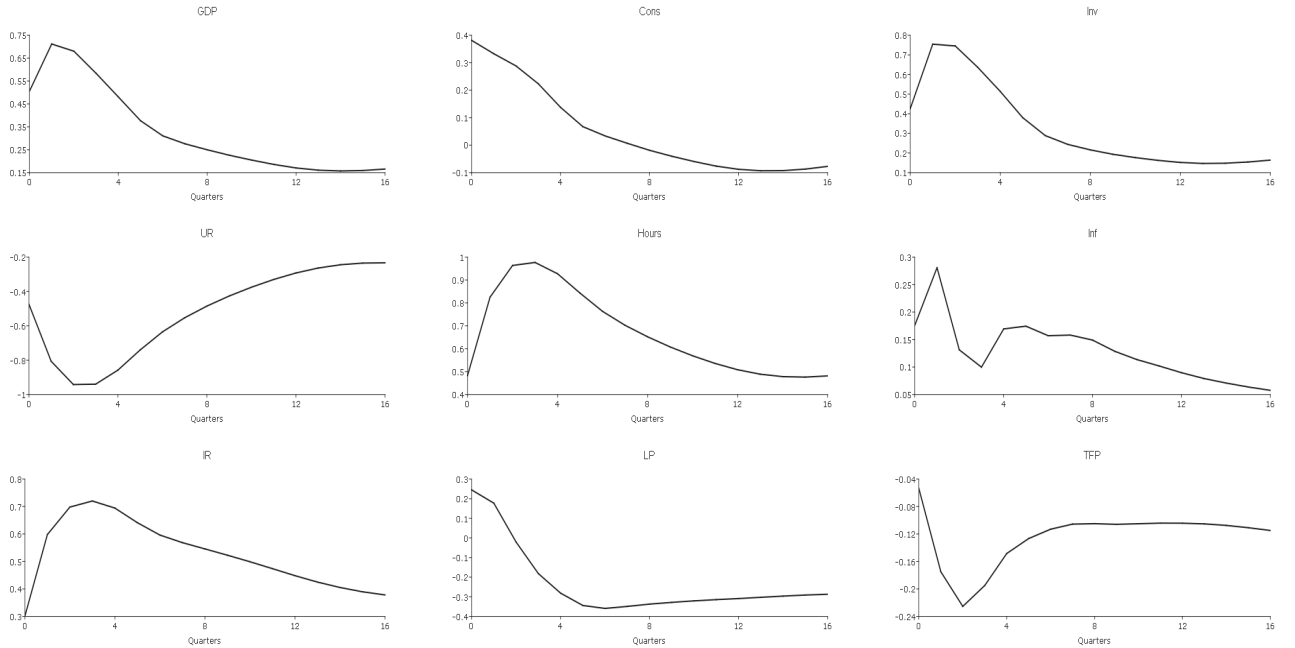


Figure 2: Cumulated impulse responses of the MBCCS

quarter delay, on Con with no delay, as well as on UR [Hours and IR] with a peak at two [three] quarters. Moreover, in view of Table 8, it explains a large fraction of the cyclical variability of Inv, Hours, UR and GDP.²¹ These results corroborate the claim that the considered shock is the main driver of the business cycle fluctuations.

However, the MBCCS has a limited positive impact on Inf, which peaks at one quarter and slowly dies out, and it marginally affects both LP and, especially, TFP. Moreover, it explains a small portion of the cyclical movements of both Inf and TFP.

Regarding the long-run scenario, the MBCCS explains around 35% of the zero-frequency variability of UR, Hours, Inv, and GDP, whereas it has negligible explanatory power for the permanent variation in Inf and TFP. Surprisingly, the MBCCS is responsible for almost the same portion of variability of Con (around 14%) both in the long and short run.²²

All in all, the findings above seem to preclude the interpretation of the MBCCS as either a productivity or a news shock on the one hand, and a traditional demand shock on the other hand. However, a meticulous interpretation of the empirical results of this application, in particular by means of a rigorous comparison with DSGE models, is beyond the scope of the present paper.

²¹In general, the contribution of the MBCCS to the cyclical variability of the various variables is smaller than those of the shocks that Angeletos *et al.* (2020) obtain by targeting UR or other procyclical variables. However, it should be reminded that the aforementioned authors make use of a VAR with 10 series, whereas in our empirical model each series is hit by 211 reduced form errors. This considerably reduces the chances that a single shock can generate a very large fraction of cyclical volatility.

²²With the important caveat in mind that the validity of bootstrap for the DRVAR needs to be formally proven, the contributions of the MBCCS to the variability of GDP, INV, UR, HOURS at periods 6-32 and ∞ are larger than twice their bootstrap standard errors. For the same variables and IR, some short horizons of their IRFs are significant at the the 68% level. Bootstrap was implemented by sampling with replacement from the DRVAR residuals.

5 Conclusions

This paper provides a link between two related but different strands of the literature on dimension reduction of multivariate time series, namely dynamic factor modelling on the one hand and the common feature methodology on the other hand. The former approach has the advantage that a limited number of factors is often enough to summarize the variation of a large dataset in economic and financial applications. However, the common feature approach has the nice theoretical property that the related common components possess a given time series feature (autocorrelation, volatility, trends, etc.) whereas the uncommon components do not.

Building on Lam *et al.* (2011) and Lam and Yao (2012), we propose a dimensional reduction approach such that both a common right space and a common left null space are present in the coefficient matrices of a large VAR model. This specification allows to detect a small dimensional VAR that is responsible for the whole dynamics of the system. This approach has many potential applications such as forecasting big data from a small scale VAR without losing relevant information, structural VAR analysis, realized covariance matrices modelling, etc.

Our Monte Carlo study shows that we should consider either the BIC or the HQIC to detect the number of the common dynamic components. We illustrate the feasibility of our framework on large dimensional macroeconomic time series relative to the US economy. Around 8 common components generate the entire dynamics of 211 aggregate economic variables. Moreover, we offer a novel approach to identify the shock that is responsible for most of the volatility of the common component at the business cycle frequency band. Such shock as a clear expansionary effect, both in the short and long run, on the labor market variables, output, and investments but it affects marginally inflation and total factor productivity.

References

- [1] AHN, S. K. (1997), Inference of Vector Autoregressive Models With Cointegration and Scalar Components, *Journal of the American Statistical Association*, 92, 350-356.
- [2] AHN, S. K. AND G. C. REINSEL (1988), Nested Reduced Rank Autoregressive Models for Multiple Time Series, *Journal of the American Statistical Association*, 83, 849-856.
- [3] ANGELETOS, G.M., COLLARD, F., AND DELLAS, H. (2020), Business-Cycle Anatomy, *American Economic Review*, 10, 3030-3070.
- [4] ATHANASOPOULOS, G., GUILLEN, O.T., ISSLER, J.V., AND VAHID, F. (2011), Model Selection, Estimation and Forecasting in VAR Models with Short-run and Long-run Restrictions, *Journal of Econometrics*, 164, 116-129.
- [5] BARSKY, R. B. AND SIMS, E. R. (2011), News shocks and business cycles, *Journal of Monetary Economics*, 58, 273-289.
- [6] BERNANKE, B., BOIVIN, J., AND P.S. ELIASZ (2005), Measuring the Effects of Monetary Policy: A Factor-augmented Vector Autoregressive (FAVAR) Approach, *The Quarterly Journal of Economics*, 120, 387-422.
- [7] BERNARDINI E. AND G. CUBADDA (2015), Macroeconomic Forecasting and Structural Analysis through Regularized Reduced-Rank Regression, *International Journal of Forecasting*, 31, 682-691.

- [8] BOIVIN, J. AND S. NG (2006), Are more data always better for factor analysis, *Journal of Econometrics*, 132, 169–194.
- [9] BRILLINGER, D. (2001), *Time series: data analysis and theory*, Society for Industrial and Applied Mathematics.
- [10] CARRIERO, A. KAPETANIOS, G., AND M. MARCELLINO (2011), Forecasting Large Datasets with Bayesian Reduced Rank Multivariate Models, *Journal of Applied Econometrics*, 26, 735-761.
- [11] CARRIERO, A. KAPETANIOS, G., AND M. MARCELLINO (2016), Structural analysis with Multivariate Autoregressive Index models, *Journal of Econometrics*, 192, 332–348.
- [12] CAVALIERE, G., L. DE ANGELIS, A. RAHBEK, AND A.M.R. TAYLOR (2015), A comparison of sequential and information-based methods for determining the co-integration rank in heteroskedastic VAR models, *Oxford Bulletin of Economics and Statistics*, 77, 106–128.
- [13] CAVALIERE, G., L. DE ANGELIS, A. RAHBEK, AND A.M.R. TAYLOR (2018), Determining the cointegration rank in heteroskedastic VAR models of unknown order, *Econometric Theory*, 34, 349–382.
- [14] CENTONI, M., AND G. CUBADDA (2003), Measuring the business cycle effects of permanent and transitory shocks in cointegrated time series. *Economics Letters* 80, 45–51.
- [15] CENTONI M., AND G. CUBADDA (2015), Common feature analysis of economic time series: An overview and recent developments, *Communications for Statistical Applications and Methods*, 22, 1–20.
- [16] CHAMBERLAIN, G., AND M. ROTHSCHILD (1983), Arbitrage, Factor Structure, and Mean-Variance Analysis on Large Asset Markets, *Econometrica*, 51, 1281-1304.
- [17] CHANG, J. Y., GUO, B., AND Q. YAO, (2018), Principal Component Analysis for Second-Order Stationary Vector Time Series, *Annals of Statistics*, 46, 2094–2124
- [18] CUBADDA, G. (2007), A Unifying Framework for Analyzing Common Cyclical Features in Cointegrated Time Series, *Computational Statistics and Data Analysis*, 52, 896–906.
- [19] CUBADDA, G., AND B. GUARDABASCIO (2019), Representation, Estimation and Forecasting of the Multivariate Index-Augmented Autoregressive Model, *International Journal of Forecasting*, 35, 67–79.
- [20] CUBADDA G., GUARDABASCIO B., AND A. HECQ (2017), A Vector Heterogeneous Autoregressive Index Model for Realized Volatility Measures, *International Journal of Forecasting*, 33, 337–344.
- [21] CUBADDA, G. AND A. HECQ (2001), On non-contemporaneous short-run comovements, *Economics Letters*, 73, 389-397.
- [22] CUBADDA, G. AND A. HECQ (2011), Testing for Common Autocorrelation in Data Rich Environments, *Journal of Forecasting*, 30, 325–335.
- [23] CUBADDA, G., AND A. HECQ (2021), Reduced Rank Regression Models in Economics and Finance, *Oxford Research Encyclopedia of Economics and Finance*, Oxford University Press, doi: 10.1093/acrefore/9780190625979.013.677, forthcoming.

- [24] CUBADDA, G., HECQ A. AND F.C. PALM (2009), Studying Co-movements in Large Multivariate Models Prior to Modeling, *Journal of Econometrics*, 148, 25-35.
- [25] DAVIS, R.A., ZANG, P. AND T. ZHENG (2016), Sparse Vector Autoregressive Modeling, *Journal of Computational and Graphical Statistics*, 25, 1077–1096.
- [26] ENGLE, R.F., AND S. KOZICKI (1993), Testing for Common Features (with comments), *Journal of Business and Economic Statistics*, 11, 369–395.
- [27] FERNALD, J.G. (2012), A Quarterly, Utilization-adjusted Series on Total Factor Productivity, *Federal Reserve Bank of San Francisco Working Paper Series*, 2012-19.
- [28] FERNÁNDEZ-VILLAVÉRDE, J., RUBIO-RAMÍREZ, J.F., AND F.SCHORFHEIDE (2016), Solution and Estimation Methods For DSGE Models, in Taylor, J.B. and H. Uhlig (eds), *Handbook of Macroeconomics*, 2, 527-724.
- [29] FORNI, M., GIANNONE, D., LIPPI, M, AND L. REICHLIN (2009), Opening the black box: structural factor models with large cross sections, *Econometric Theory*, 25, 1319–1347.
- [30] FRANCIS, N., OWYANG, M. T., ROUSH, J. E., AND DICECIO, R. (2014), A Flexible Finite-Horizon Alternative to Long-run restrictions with an application to technology shocks, *Review of Economic Statistics*, 96, 638-647.
- [31] GOETZ, T., HECQ, A. AND S. SMEEKES (2016), Testing for Granger-causality in large mixed-frequency VARs, *Journal of Econometrics*, 193, (2), 418-432.
- [32] HECQ, A., PALM, F.C. AND J.P. URBAIN (2006), Common cyclical features analysis in VAR models with cointegration, *Journal of Econometrics*, 132, 117–141.
- [33] HECQ, A., MARGARITELLA, L. AND S. SMEEKES (2021), Granger Causality Testing in High-Dimensional VARs: a Post-Double-Selection Procedure, *Journal of Financial Econometrics*, forthcoming.
- [34] HSU, N. J. AND H. L. HUNG AND Y. M. CHANG (2008), Subset selection for vector autoregressive processes using Lasso, *Computational Statistics & Data Analysis*, 52, 3645-3657.
- [35] HU, Z., DONG, K., DAI, W. AND T. TONG (2017), A Comparison of Methods for Estimating the Determinant of High-Dimensional Covariance Matrix, *International Journal of Biostatistics*, 13, 1-24.
- [36] KAPETANIOS, G. (2004), The asymptotic distribution of the cointegration rank estimator under the Akaike information criterion, *Econometric Theory*, 20, 735–742.
- [37] KARLSSON, S. (2013), Forecasting with Bayesian Vector Autoregression, *Handbook of Economic Forecasting*, 2B, 791-897, North Holland.
- [38] KOCK, A. B. AND CALLOT, L. (2015), Oracle inequalities for high dimensional vector autoregressions, *Journal of Econometrics*, 186, 325–344.
- [39] KOHN, R. (1982), When is an aggregate of a time series efficiently forecast by its past?, *Journal of Econometrics*, 18, 337-349.

- [40] KOOP, G. (2017), Bayesian Methods for Empirical Macroeconomics with Big Data, *Review of Economic Analysis*, 9 , 33-56.
- [41] LAM, C. AND Q. YAO (2012), Factor modeling for high-dimensional time series: Inference for the number of factors, *Annals of Statistics*, 40, 694-726.
- [42] LAM, C., YAO, Q., AND N. BATHIA, (2011), Estimation of latent factors for high-dimensional time series, *Biometrika*, 98 901–918.
- [43] LI, W., GAO, J., LI K., AND Q. YAO (2016), Modeling Multivariate Volatilities via Latent Common Factors, *Journal of Business & Economic Statistics*, 34, 564-573.
- [44] LI, Z., WANG, Q.W., AND J. YAO (2017), Identifying the number of factors from singular values of a large sample auto-covariance matrix, *Annals of Statistics*, 45, 257–288.
- [45] LIPPI, M. (2018), Frequency-Domain Approach in High-Dimensional Dynamic Factor Models, *Oxford Research Encyclopedia of Economics and Finance*, Oxford University Press, doi: 10.1093/acrefore/9780190625979.013.171.
- [46] LIPPI, M. (2019), Time-Domain Approach in High-Dimensional Dynamic Factor Models, *Oxford Research Encyclopedia of Economics and Finance*, Oxford University Press, doi: 10.1093/acrefore/9780190625979.013.169.
- [47] LÜTKEPOHL, H (1984A), Forecasting Contemporaneously Aggregated Vector ARMA Processes, *Journal of Business and Economic Statistics*, 2, 201-214.
- [48] LÜTKEPOHL, H (1984B), Linear transformations of vector ARMA processes, *Journal of Econometrics*, 26, 283-293.
- [49] LÜTKEPOHL, H (2005), *New Introduction to Multiple Time Series Analysis*, Springer-Verlag Berlin.
- [50] MCCracken, M.W. AND NG, S., (2020), FRED-QD: A Quarterly Database for Macroeconomic Research, *Federal Reserve Bank of St. Louis Working Paper*, 2020-005, <https://doi.org/10.20955/wp.2020.005>.
- [51] NICHOLSON, W., WILMS, I., BIEN, J., AND D.S. MATTESON (2018), High Dimensional Forecasting via Interpretable Vector Autoregression, *arXiv:1412.5250v3*.
- [52] NIELSEN, B. (2006), Order determination in general vector autoregressions, *IMS Lecture Notes - Monograph Series*, 52, 93–112.
- [53] OBERHOFER, W. AND J. KMENTA (1974), A general procedure for obtaining maximum likelihood estimates in generalized regression models, *Econometrica*, 42, 579–590.
- [54] PESARAN, M.H., PICK, A., AND A. TIMMERMANN (2011), Variable selection, estimation and inference for multi-period forecasting problems, *Journal of Econometrics*, 164, 173–187.
- [55] QUENOUILLE, M.H. (1957), *The Analysis of Multiple Time Series*, Griffin’s Statistical Monographs & courses.
- [56] REINSEL, G., (1983), Some results on multivariate autoregressive index models, *Biometrika*, 70 (1), 145–156.

- [57] SIMS, C. (1980), Macroeconomics and Reality, *Econometrica*, 48, 1-48.
- [58] SMEEKES, S. AND E. WIJLER, E. (2018), Macroeconomic forecasting using penalized regression methods, *International Journal of Forecasting*, 34, 408-430.
- [59] STOCK, J. H. AND WATSON, M. W. (2016), Dynamic Factor Models, Factor-Augmented Vector Autoregressions, and Structural Vector Autoregressions in Macroeconomics, in Taylor, J.B. and H. Uhlig (eds), *Handbook of Macroeconomics*, Vol. 2, North Holland.
- [60] TAO, M., WANG, Y, YAO, Q., AND J ZOU (2011), Large Volatility Matrix Inference via Combining Low-Frequency and High-Frequency Approaches, *Journal of the American Statistical Association*, 106, 1025-1040.
- [61] UHLIG, H. (2003), What moves real GNP? *mimeo HU Berlin*.
- [62] VAHID, F., AND R.F. ENGLE (1993), Common trends and common cycles, *Journal of Applied Econometrics*, 8, 341-360.
- [63] VELU, R., REINSEL, G., AND D. WICHERN (1986), Reduced Rank Models for Multiple Time Series, *Biometrika*, 73, 105-118.
- [64] WILMS, I. AND C. CROUX (2016), Forecasting using sparse cointegration, *International Journal of Forecasting*, 32(4), 1256-1267.
- [65] ZHANG, R., ROBINSON, P., AND Q. YAO (2019), Identifying Cointegration by Eigenanalysis, *Journal of the American Statistical Association*, 114, 916-927.

6 Appendix

In this appendix we prove the main results of Subsection 2.2. and provide the details about the key variables that are analyzed in Section 4.

6.1 Mean square errors of the OLS and GLS estimators

In this subsection we provide the proof of Theorem 6. The thesis is given by the following inequality:

$$E[\text{Vec}(\hat{\alpha} - \alpha)\text{Vec}(\hat{\alpha} - \alpha)'|Z] \geq E[\text{Vec}(\tilde{\alpha} - \alpha)\text{Vec}(\tilde{\alpha} - \alpha)'|Z] \quad (22)$$

for $\forall \text{Vec}(\alpha) \in \mathbb{R}^{nr}$.

Let us start with inserting Equation (13) into (15) and (17) to respectively get

$$\text{Vec}(\hat{\alpha}) = \text{Vec}(\alpha) + [A' \otimes (Z'Z)^{-1}Z']\text{Vec}(u) \quad (23)$$

$$\text{Vec}(\tilde{\alpha}) = \text{Vec}(\alpha) + \left[(A'\Sigma_u^{-1}A)^{-1}A'\Sigma_u^{-1} \otimes (Z'Z)^{-1}Z' \right] \text{Vec}(u) \quad (24)$$

In view of Equations (23) and (24) we get

$$E[\text{Vec}(\hat{\alpha} - \alpha)\text{Vec}(\hat{\alpha} - \alpha)'|Z] = A'\Sigma_u A \otimes (Z'Z)^{-1}$$

and

$$E[\text{Vec}(\tilde{\alpha} - \alpha)\text{Vec}(\tilde{\alpha} - \alpha)'|Z] = (A'\Sigma_u^{-1}A)^{-1} \otimes (Z'Z)^{-1}$$

Since the Kronecker product of two positive semidefinite matrix is positive semidefinite as well, proving (22) requires to show that

$$A'\Sigma_u A \geq (A'\Sigma_u^{-1}A)^{-1} \quad (25)$$

In order to prove inequality (25), we notice that

$$B'\Sigma_u B = (B'\Sigma_u^{-1}B)^{-1}$$

where $B = [A, A_\perp]$. Partitioning conformably in blocks both sides of the equation above we get

$$\begin{bmatrix} A'\Sigma_u A & A'\Sigma_u A_\perp \\ A'_\perp \Sigma_u A & A'_\perp \Sigma_u A_\perp \end{bmatrix} = \begin{bmatrix} A'\Sigma_u^{-1}A & A'\Sigma_u^{-1}A_\perp \\ A'_\perp \Sigma_u^{-1}A & A'_\perp \Sigma_u^{-1}A_\perp \end{bmatrix}^{-1} \quad (26)$$

Applying the rule of the inverse of a partitioned symmetric matrix to the upper left block of the matrices in (26) we see

$$A'\Sigma_u A = (A'\Sigma_u^{-1}A)^{-1} + (A'\Sigma_u^{-1}A)^{-1} A'\Sigma_u^{-1}A_\perp [A'_\perp \Sigma_u^{-1}A_\perp - A'_\perp \Sigma_u^{-1}A(A'\Sigma_u^{-1}A)^{-1}A'\Sigma_u^{-1}A_\perp]^{-1} A'_\perp \Sigma_u^{-1}A(A'\Sigma_u^{-1}A)^{-1} \quad (27)$$

Finally, noticing that the matrix in square brackets in (27) is the conditional variance matrix of $A'_\perp \Sigma_u^{-1}u_t$ given $A'\Sigma_u^{-1}u_t$, then inequality (25) trivially follows.

Moreover, the equality sign holds when $A'\Sigma_u^{-1}A_\perp = 0$, that is when the matrix A belongs to the space spanned by r of the eigenvectors of Σ_u^{-1} , and consequently A_\perp belongs to the space spanned by the remaining $n - r$ eigenvectors. Since the eigenvectors of Σ_u^{-1} are the same of Σ_u , we notice that that two estimators have the same variance when $A'u_t$ and $A'_\perp u_t$ are not correlated.

6.2 Consistency of the BIC and HQIC for r .

In this subsection we show that the BIC and HQIC provide weakly consistent estimators for the number of dynamic components r but not for the overall number of DRVAR parameters k . We start by reviewing the conditions on the penalty term c_T for weak consistency of an information criterion such as (18) when n is fixed and T diverges: $c_T \rightarrow \infty$ and $\frac{c_T}{T} \rightarrow 0$ as $T \rightarrow \infty$, see e.g. Lütkepohl (2005). It easily follows that BIC and HQIC are consistent for k whereas the AIC is not when n is finite.

Let us assume that both OLS and FGLS estimate the DRVAR parameters (up to an orthonormal transformation) with the standard \sqrt{T} rate as $n, T \rightarrow \infty$, and that $\gamma = \lim_{n \rightarrow \infty} \left(\prod_{i=1}^n \sigma_i^2 \right)^{1/n}$ exists. When $q \geq r$, $\hat{\sigma}_i^2(q)$ is a \sqrt{T} -consistent estimator of σ_i^2 for any i . When $q < r$, $\text{plim}_{n, T \rightarrow \infty} \hat{\sigma}_i^2(q) \geq \sigma_i^2$ for any i and, given that the factors are assumed to be strong, the case where each element of \bar{a}_j is $O(1)$ for $j = 1, \dots, r$ is included, implying that the dynamic component x_t influences most of the n time series. Hence, it follows that

$$\text{plim}_{n, T \rightarrow \infty} \hat{\gamma}_q = \begin{cases} \gamma & \text{for } q \geq r \\ \gamma^* > \gamma & \text{for } q < r \end{cases}$$

where $\hat{\gamma}_q = \left(\prod_{i=1}^n \hat{\sigma}_i^2 \right)^{1/n}$.

Let us rewrite Equation (18) as

$$\text{IC}(q) = \ln(\hat{\gamma}_q) + \frac{c_T k}{Tn}$$

Since

$$\lim_{n, T \rightarrow \infty} \frac{k}{n} = \lim_{n \rightarrow \infty} \frac{nq + (p-1)q^2}{n} = q$$

we conclude that $\text{IC}(q)$ is asymptotically equivalent to

$$\text{IC}^*(q) = \ln(\hat{\gamma}_q) + \frac{c_T}{T}q,$$

and that $\text{IC}^*(q)$ is weakly consistent for r (but not for k) when the penalty term c_T is the one of the HQIC or BIC.

6.3 Key variables and their transformations

We report in Table 9 the denominations of the nine key variables that are analyzed in Section 4 along with the relative transformations.

Table 9: Variables and their transformations

Variable	Transformation	Acronym
Real Gross Domestic Product	$(1 - L) \log$	GDP
Real Personal Consumption Expenditures	$(1 - L) \log$	Con
Real Gross Private Domestic Investment	$(1 - L) \log$	Inv
Civilian Unemployment Rate	$(1 - L)$	UR
Nonfarm Business Sector: Hours of All Persons	$(1 - L) \log$	Hours
Personal Consumption Expenditures: Chain-type Price Index	$(1 - L)^2 \log$	Inf
Effective Federal Funds	$(1 - L)$	IR
Nonfarm Business Sector: Real Output Per Hour of All Persons	$(1 - L) \log$	LP
Total Factor Productivity	$(1 - L) \log$	TFP

RECENT PUBLICATIONS BY *CEIS Tor Vergata*

Child labour theories and policies

Furio Camillo Rosati

CEIS Research Paper, 533, March 2022

Indirect Savings from Public Procurement Centralization

Clarissa Lotti and Giancarlo Spagnolo

CEIS Research Paper, 532, February 2022

Who cares when Value (Mis)reporting May Be Found Out?

An Acquiring-a-Company Experiment with Value Messages and Information Leaks

Daniela Di Cagno, Werner Güth, Tim Lohse, Francesca Marazzi and Lorenzo Spadoni

CEIS Research Paper, 531, January 2022

Relational Skills and Corporate Productivity

Leonardo Becchetti, Sara Mancini and Nazaria Solferino

CEIS Research Paper, 530, January 2022

Born to Run: Adaptive and Strategic Behavior in Experimental Bank-Run Games

Federico Belotti, Eloisa Campioni, Vittorio Larocca, Francesca Marazzi, Luca Panaccione and Andrea Piano Mortari

CEIS Research Paper, 529, December 2021

The Effect of Mandatory Non-financial Reporting on CSR (and Environmentally Sustainable) Investment: a Discontinuity Design Approach

Leonardo Becchetti, Sara Mancini and Nazaria Solferino

CEIS Research Paper, 528, November 2021

On Some Problems of Using the Human Development Index in Economic History

Nicola Amendola, Giacomo Gabbuti and Giovanni Vecchi

CEIS Research Paper, 527, November 2021

Banking Diversity, Financial Complexity and Resilience to Financial Shocks: Evidence From Italian Provinces

Beniamino Pisicoli

CEIS Research Paper, 526, November 2021

Reduced Rank Regression Models in Economics and Finance

Gianluca Cubadda and Alain Hecq

CEIS Research Paper, 525, November 2021

Herding and Anti-Herding Across ESG Funds

Rocco Ciciretti, Ambrogio D'Alò and Giovanni Ferri

CEIS Research Paper, 524, November 2021

DISTRIBUTION

Our publications are available online at www.ceistorvergata.it

DISCLAIMER

The opinions expressed in these publications are the authors' alone and therefore do not necessarily reflect the opinions of the supporters, staff, or boards of CEIS Tor Vergata.

COPYRIGHT

Copyright © 2022 by authors. All rights reserved. No part of this publication may be reproduced in any manner whatsoever without written permission except in the case of brief passages quoted in critical articles and reviews.

MEDIA INQUIRIES AND INFORMATION

For media inquiries, please contact Barbara Piazzi at +39 06 72595652/01 or by e-mail at piazzi@ceis.uniroma2.it. Our web site, www.ceistorvergata.it, contains more information about Center's events, publications, and staff.

DEVELOPMENT AND SUPPORT

For information about contributing to CEIS Tor Vergata, please contact at +39 06 72595601 or by e-mail at segr.ceis@economia.uniroma2.it