CEIS TOR VERGATA Research Papers

Università di Roma Tor Vergata

ISSN 2610-931X

CEIS Tor Vergata

RESEARCH PAPER SERIES

Vol. 20, Issue 6, No. 547 – December 2022

Testing for Endogeneity of Irregular Sampling Schemes

Aleksey Kolokolov, Giulia Livieri and Davide Pirino

Testing for endogeneity of irregular sampling schemes*

Aleksey Kolokolov† Giulia Livieri‡ Davide Pirino§

December 2, 2022

Abstract

In the context of high-frequency financial data it is often assumed that sampling times are exogenous. This entails that financial asset prices, sampled on a grid of trade instants, are independent from the sampling times. We derive statistical tests capable of determining whether or not, and to what extent, this hypothesis is rejected by the data. We test for sampling time endogeneity in relation to both the efficient and the noise components of the observed price. Using a vast dataset of financial asset prices we give empirical evidence that the efficient component of the observed price process does not show a dependence with trade arrival times of the kind that may jeopardize well-known results on convergence of power variations. In addition, we provide empirical evidence that the assumption of independence between market microstructure noise and trading instants is not supported by the data.

Keywords: irregular sampling, sampling schemes, zeros, power variation.

*Davide Pirino acknowledges support from the project HiDEA (Advanced Econometric methods for High-frequency Data) financed by the Italian Ministry of Education, University and Research under the program PRIN 2017, Prot. 2017RSMPZZ.

†*Affiliation:* Alliance Manchester Business School, The University of Manchester, Booth St W, Manchester M15 6PB, UK. *E-mail:* alexeiuo@gmail.com.

‡*Affiliation:* Scuola Normale Superiore, Piazza dei Cavalieri 7, 56123, Pisa, Italy. *E-mail:* giulia.livieri@sns.it.

§*Affiliation:* Università degli Studi di Roma "Tor Vergata", Dipartimento di Economia e Finanza, Via Columbia 2, 00173, Roma. *E-mail:* davide.pirino@gmail.com.

1 Introduction

Over the last decades financial econometrics has made a substantial progress in the analysis of high-frequency data. Asset prices recorded at a high-frequency are interpreted as the values of a semimartingale X observed at N_t^n points of a grid $\pi_n = \{t(n,i) \mid i = 0,\ldots,N_t^n\}$, which form a partition of a fixed time interval $[0,t]$. In this context, objects of interest are functionals of the form:

$$\mathsf{PV}\left(X,f\right)_n = \sum_{i=1}^{N_t^n} \Delta\left(n,i\right)^{1-r/2} f\left(\Delta_i^n X\right), \tag{1}$$

where PV stands for Power Variation, $\Delta_i^n X \doteq X_{t(n,i)} - X_{t(n,i-1)}$ denotes the increment of X over the interval $[t(n,i), t(n,i-1)]$, $\Delta\left(n,i\right) \doteq t(n,i) - t(n,i-1)$, and either $f\left(x\right) = |x|^r$ or $f\left(x\right) = x^r$, $r > 0$. The limiting behaviour (as $n \to \infty$) of $\mathsf{PV}\left(X,f\right)_n$ is well-studied under the assumption that the sampling times are independent from X, in particular, when the partition π_n is deterministic (see, among many others, Jacod (2008); Barndorff-Nielsen et al. (2006); Barndorff-Nielsen and Shephard (2002), for the equidistant case and Mykland and Zhang (2006); Barndorff-Nielsen and Shephard (2006), for the non-equidistant one). A general treatment of the matter is given by Hayashi et al. (2011). For a wide class of sampling schemes, when X is a continuous Itō semimartingale, independent from the sampling times $t(n,i)$'s, with quadratic variation $\int_0^t \sigma_s^2\, ds$, we have:

$$\mathsf{PV}\left(X,f\right)_n \overset{u.c.p.}{\longrightarrow} \mu_r \int_0^t f\left(\sigma_s\right)\, ds, \tag{2}$$

where $\mu_r = \mathbb{E}\left[f(u)\right]$ with u a standard normal random variable and "$\overset{u.c.p.}{\longrightarrow}$" indicates the uniform convergence in probability over $[0,t]$. The above convergence is crucial for the inference based on power variation. However, it can be violated due to the dependence between X and the sampling times. In what follows, we say that a sampling scheme is exogenous if the convergence in equation (2) is preserved for all regular enough f, and it is endogenous otherwise[1]. That is, we informally use the term endogeneity to indicate sampling schemes for which the results obtained under the independence between X and π_n do not hold. Whether or not real financial data show such endogeneity (and of what kind) remains an open question.

In this paper, we tackle the problem of testing for endogeneity in the presence of market microstructure noise and jumps. First, we propose a test for endogeneity in the absence of noise. Second, we investigate the problem of detecting time endogeneity when the observations of X are contaminated by a noise. In this framework, we propose two complementary tests: a test for the endogeneity of the sampling times with respect to the microstructure noise and a robust-to-noise test for endogeneity of the sampling times with respect to the efficient price.

[1]Endogeneity of sampling times, in its most general meaning, may stand for any kind of dependence between the sampling times in π_n and the sampled process X. It can be defined in multiple ways (see, for example, Fukasawa, 2010a,b). In this paper we only focus on detecting "strong" forms of dependence, i.e. ones that jeopardize standard results for power variation. We leave investigating other forms of dependence between the sampling times and X for future research.

In a related work, Li et al. (2014) propose an alternative test for endogeneity under the absence of microstructure noise based on the realized tricity $\mathsf{PV}\left(X, x^3\right)_n$. Their test detects a special kind of endogeneity, one which implies that $\mathsf{PV}\left(X, x^3\right)_n$ converges to a nonzero limit in probability, whereas, under the independence between X and π_n, the realized tricity converges to zero. Nevertheless, if the sampling times are successive hitting times of a symmetric spatial grid by X, the tricity still converges to zero. Accordingly, the Li et al. (2014)'s test has zero or near zero power in this case. On the other hand, our approach allows to detect endogeneity for any configuration of the barriers. This is achieved due to a different testing principle: our test is based on comparing two types of perturbed power variations computed using the increments of the observed process corresponding to longer and shorter durations of time. The null of non-endogeneity is rejected anytime that the difference between the two kinds of power variations is asymptotically different from zero. This may occur either with a non-zero covariance between $|\Delta_i^n X|$ and the corresponding time increments $\Delta\left(n, i\right)$, or in cases, such as the successive hitting times of a symmetric spatial grid by X, in which that covariance is exactly zero.

To the best of our knowledge, our proposed endogeneity tests under the presence of noise have no analogous in the existing literature. We observe $Y \doteq X + U$ in the noisy case, where X is the efficient price and U is a zero-mean error sequence. Both components of the observed process can be potentially dependent on sampling times. Moreover, they can be correlated with each other. Thus, testing the endogeneity of a sampling schemes is much more challenging than in the noiseless setting. At the cost of imposing an additional structural assumption on the sampling times (which is still a nonparametric assumption accommodating a wide range of sampling schemes), we reduce the problem of testing for endogeneity to a problem of testing the significance of a coefficient in a semiparametric regression with time-varying intercept. The inference of time-varying regression models has been extensively studied in the literature (Gao and Hawthorne, 2006; Zhang and Wu, 2012; Kalli and Griffin, 2014; Vogt, 2015; Zhang and Wu, 2015). Typically the varying coefficients are assumed to be smooth functions of time. We build upon this literature by deriving a significance test assuming that the varying coefficient is a realization of a semimartingle (as implied by our assumption on the sampling times). As a result we first construct a test for independence between the noise and the sampling times, robust to efficient price-noise dependence.

Secondly, we show that it is possible to profit from the highest sampling frequency to test for dependence between the efficient component of Y and the sampling times. To do so, we rely on the pre-averaging method that has been successfully applied to remove the impact of noise in estimating efficient price characteristics (see Jacod et al., 2009; Podolskij and Vetter, 2009a,b; Christensen et al., 2010; Jacod et al., 2010, among many others). We construct the pre-averaged analog of the statistic used to detect the dependence between the noise and the sampling time. We show that, under the null of a sampling process independent from the efficient price X, the statistics based on pre-averaged quantities is distributed, asymptotically, as a standard Gaussian. Notably, this result is robust to time-noise dependence, an important feature in light of our empirical findings.

Our theoretical results contribute to an extensive literature that has investigated, under different aspects, the dependence between prices of financial assets and times at which they are observed (see Oomen, 2006; Fukasawa, 2010b; Hayashi et al., 2011; Fukasawa and Rosenbaum, 2012; Li et al., 2013, 2014; Bibinger and Mykland, 2016; Potiron and Mykland, 2017; Cui, 2021; Dimitriadis and Halbleib, 2022; Merrick and Linton, 2022, among others). The majority of these studies derive estimators of price characteristics (e.g. quadratic variation) robust to different specifications of time endogeneity, which the proposed theory allows to test.

As an empirical application, using a vast dataset of financial asset prices, we provide statistically robust empirical evidence that I) the efficient component of the observed price shows no dependence from the trade arrival times; II) the opposite occurs for microstructure noise, whose dependence from trade times is striking; III) for two sampling schemes, namely business time and dollar-volume sampling, featuring moderate sampling frequencies (such as one observation, on average, every minute or more) the observed price (now assumed noiseless) depends on the sampling times.

The paper has the following structure. We begin with a general discussion, presented in Section 2, on the main assumptions that are used to derive the central limit theorems. We present a test for endogenous time in the absence of microstructure noise in Section 3. In Subsection 3.2 we discuss the impact of jumps and we provide a version of the test which is robust to discontinuities. In Section 4 we study time endogeneity in presence of microstructure noise. More specifically, we discuss how to identify a dependence between sampling times and microstructure noise in Subsection 4.1 and between the efficient component of the observed price (i.e. the observed price purged from the microstructure noise contamination) and the sampling times in Subsection 4.2. A Monte Carlo assessment of the finite sample properties of the proposed asymptotic theory is discussed in Section 5. In Section 6 we apply the newly derived tests to a vast dataset of NYSE stocks. To conclude, we summarize our research in Section 7. Finally, all proofs and technical lemmas are reported in Appendix A.

2 Settings and hypothesis

We begin with general conditions imposed, throughout the paper, on the efficient log-price process (Assumption $\mathcal{A}_1$), its volatility (Assumption $\mathcal{A}_2$) and the sampling schemes (Assumption $\mathcal{A}_3$). In what follows, we assume the existence of a rich enough filtered probability space $(\Omega, (\mathcal{F}_t)_{t\geq 0}, \mathcal{F}, \mathbb{P})$ satisfying usual conditions (Jacod and Protter, 2004). We derive our asymptotic theory by letting the number of observations go to infinity over a fixed time horizon $[0, t]$, with $t \leq 1$ (the value $t = 1$ representing, as it is customary, a trading day). For this reason we also assume that $\mathcal{F}_1 = \mathcal{F}$.

$\mathcal{A}_1$ The real-valued logarithmic efficient price process $\{X_t; t \geq 0\}$ is an Itō semimartingale

$$X_t = X_0 + \int_0^t \mu_s \, ds + \int_0^t \sigma_s \, dW_s, \tag{3}$$

where W_s is a Brownian motion. The process μ_s is a locally bounded process and the process σ_s satisfies Assumption $\mathcal{A}_2$ below.

$\mathcal{A}_2$ The volatility process is a possibly discontinuous Itō semimartingale, which can be written as

$$\sigma_t = \sigma_0 + \int_0^t \widetilde{\mu}_s \, ds + \int_0^t \widetilde{\sigma}_s \, dW_s + M_t + \sum_{s \leq t} \Delta\sigma_s \mathbf{1}_{\{|\Delta\sigma_s|>1\}}, \tag{4}$$

where M is a local martingale with $|\Delta M_t| \leq 1$, orthogonal to W and its predictable quadratic variation process is $\langle M, M \rangle_t = \int_0^t a'_s \, ds$. The predictable compensator of $\sum_{s \leq t} \Delta\sigma_s \mathbf{1}_{\{|\Delta\sigma_s|>1\}}$ is $\int_0^t a_s \, ds$. Moreover, the processes a, a', $\widetilde{\mu}$ are locally bounded, and $\widetilde{\sigma}$ is left continuous with right limits.

$\mathcal{A}_3$ At the sampling frequency characterized by the integer n, the process $\{X_t; t \geq 0\}$ is observed along a strictly increasing sequence of finite (possible random) times $t(n,i), i \geq 0$, with $t(n,0) = 0$. Setting

$$\begin{aligned} \Delta(n,i) &= t(n,i) - t(n,i-1), \qquad & I(n,i) &= (t(n,i-1), t(n,i)] \\ N_t^n &= \inf\,(i \,:\, t(n,i) > t) - 1, \qquad & |\pi_t^n| &= \sup_{i=1,\ldots,N_t^n+1} \Delta(n,i), \end{aligned}$$

the following properties are always assumed to hold:

$$\begin{aligned} &\forall n \geq 1 \;\Rightarrow\; t(n,i) \to \infty, \qquad \mathbb{P}-a.s., \;\; as \;\; i \to \infty, \\ &\forall t \geq 0 \;\Rightarrow\; |\pi_t^n| \xrightarrow{p} 0, \qquad as \;\; n \to \infty. \end{aligned}$$

Assumption $\mathcal{A}_1$ is ubiquitous in continuous-time financial econometrics. For sake of exposition, we do not include a discontinuous component in X. However, we show that our proposed test is robust to jumps in Subsection 3.2. Assumption $\mathcal{A}_2$ allows for jumps in volatility and the leverage effect. The local martingale M may have jumps and a non-vanishing continuous martingale part, which can be expressed as an integral with respect to a Brownian motion independent from W. Assumption $\mathcal{A}_3$ imposes certain asymptotic regularity on the sampling schemes: time duration between observations converges in probability to zero as $n \to \infty$, but for every fixed n the sampling times are sufficiently distant from each other.

With the above assumptions at hand we can formally state the hypothesis to be tested. As pointed out by Li et al. (2014), the crucial condition on the dependence between X and $\{t(n,i)\}$, which allows to derive equation (2), is that the moments of the rescaled increments of the Brownian component of X (i.e. $\Delta(n,i)^{-1/2}\Delta_i^n W$) coincide with the moments of standard normal distribution.

Therefore, in order to test for endogeneity in the absence of microstructure noise, we consider the following null hypothesis:

$$\mathcal{H}_0: \quad \mathbb{E}\left[\left|\Delta(n,i)^{-1/2}\Delta_i^n W\right|^r \,\middle|\, \Delta(n,i)\right] = \mu_r, \quad \forall r > 0,\ i = 1, \ldots, N_t^n.$$

The null $\mathcal{H}_0$ holds for all sampling times independent of X, in particular, for all nonrandom but irregularly spaced times. A paradigmatic example of the violation of $\mathcal{H}_0$ is when the sampling times are generated by hitting a barrier as in the assumption below.

$\mathcal{H}_1$ The sequence of sampling times $\{t(n,i)\}$ is defined recursively: $t(n,0) = 0$ and $t(n,i+1)$ is the first time $t > t(n,i)$ so that $X_t - X_{t(n,i)}$ is either larger than $n^{1/2}a$ or smaller than $-n^{1/2}b$, for every $i > 0$ and some $a, b > 0$.

Assumption $\mathcal{H}_1$ is the reference example of endogenous sampling used throughout the paper. When $a = b$, for a sufficiently large n, the distribution of $\Delta_i^n X$ is symmetric around zero. In this case, we call sampling schemes satisfying Assumption $\mathcal{H}_1$ symmetric barrier hitting sampling.

A market microstructure noise heavily contaminates real financial data sampled at ultra high-frequency. In this framework, instead of observing a discretization of X, at times $t(n,i)$ we assume to record the values of a process Y contaminated by a microstructure noise as defined in the assumption below.

$\mathcal{D}_1$ The observed log-price process $\{Y_t; t \geq 0\}$ has the following form:

$$Y_{t(n,i)} = X_{t(n,i)} + U_{t(n,i)}, \tag{5}$$

where the process X is defined in Assumption $\mathcal{A}_1$ and $\left\{U_{t(n,i)}\right\}$ is a double-sequence of zero-mean random variables.

The addition of noise makes the logic of our approach more compelling. Indeed, both components of the observed price process can, in principle, depend on sampling times. However, the increments of the efficient price are not observed directly. We use pre-averaging to eliminate the effect of noise, which makes testing $\mathcal{H}_0$ problematic. Indeed, pre-averaged increments of X include high-frequency returns corresponding to different time durations. Therefore, to test the endogeneity in the noisy setting, we consider a different (stronger) null hypothesis instead of $\mathcal{H}_0$. For simplicity, we choose the null hypothesis to be the independence between the (increments of the) sampling times and the (increments of the) efficient price X:

$$\mathcal{H}_{X\perp\Delta}: \ \Delta_i^n X \text{ are independent from } \Delta(n,i) \text{ for all } n \text{ and every } i.$$

We aim at testing $\mathcal{H}_{X\perp\Delta}$ against a barrier hitting sampling scheme (symmetric or asymmetric). Our proposed test is based on the observation that under $\mathcal{H}_{X\perp\Delta}$ the pre-averaged increments of

X are uncorrelated with their time durations. Any dependence between X and π_n preserving this property can not be detected. We do not intend to detect such kinds of endogeneity and leave them for future research.

The null hypothesis for testing for endogeneity of the sampling times with respect to the noise sequence is defined analogously:

$$\mathcal{H}_{U\perp\Delta}: \ \Delta_i^n U \text{ are independent from } \Delta(n,i) \text{ for all } n \text{ and every } i.$$

In what follows we propose a regression-based approach to test $\mathcal{H}_{U\perp\Delta}$.

3 Endogenous sampling times: the frictionless case

3.1 The test

We propose to test $\mathcal{H}_0$ by evaluating the difference between two forms of randomly perturbed realized power variations. The proposed test statistic takes the form:

$$\mathsf{T}\left(X,\{t(n,i)\}\right) = \frac{\sum_{i=1}^{N_t^n-1} \Delta(n,i)\left(\Delta(n,i)^{-\frac{r}{2}}\left|\Delta_i^n X\right|^r - \Delta(n,i+1)^{-\frac{r}{2}}\left|\Delta_{i+1}^n X\right|^r\right)\alpha(n,i)}{\sqrt{\frac{2(\mu_{2r}-\mu_r^2)}{\mu_{2r}} \sum_{i=1}^{N_t^n-1} \Delta(n,i)^{2-r}\left|\Delta_i^n X\right|^{2r}\alpha(n,i)}}, \tag{6}$$

where $r>0$ and $\mu_r = \mathbb{E}\left[|u|^r\right]$, with u is a standard normal random variable, and

$$\alpha(n,i) = \begin{cases} \mathbf{1}_{\{\Delta(n,i)>\Delta(n,i+1)\}}, & i=1, \\ \mathbf{1}_{\{\Delta(n,i)>\Delta(n,i+1)\}}\mathbf{1}_{\{\Delta(n,i-1)\leq\Delta(n,i)\}}, & i\geq 2, \end{cases} \tag{7}$$

where $\mathbf{1}_A$ is the indicator function of a generic event $A\in\mathcal{F}$. Intuitively, the test statistic compares the two types of perturbed power variations defined as:

$$\mathsf{V}_t^n = \sum_{i=1}^{N_t^n-1} \Delta(n,i)^{1-\frac{r}{2}}\left|\Delta_i^n X\right|^r \alpha(n,i), \quad \mathsf{V}_t'^n = \sum_{i=1}^{N_t^n-1} \Delta(n,i)\Delta(n,i+1)^{-\frac{r}{2}}\left|\Delta_{i+1}^n X\right|^r \alpha(n,i),$$

which converge to the same limit under $\mathcal{H}_0$. Indeed, as follows from the proof of Theorem 3.1 below, under $\mathcal{H}_0$, both V_t^n and $\mathsf{V}_t'^n$ converge uniformly in probability to $\mu_r\int_0^t|\sigma_s|^r\tilde{a}(1)_s\,ds$, where $\tilde{a}(1)_s$ is a stochastic process defined below. Hence, the difference $\mathsf{V}_t^n - \mathsf{V}_t'^n$ converges to zero. The denominator of $\mathsf{T}\left(X,\{t(n,i)\}\right)$ standardizes the difference by an estimator of its asymptotic standard deviation. On the other hand, under endogeneity, the difference $\mathsf{V}_t^n - \mathsf{V}_t'^n$ does not converge to zero. For example, if $\Delta(n,i)$s are positively correlated with the $|\Delta_i^n X|$s, the event $\{\alpha(n,i)=1\}$ indicates that the rescaled increments $\Delta(n,i)^{-\frac{r}{2}}\left|\Delta_i^n X\right|^r$ are likely larger than $\Delta(n,i+1)^{-\frac{r}{2}}\left|\Delta_{i+1}^n X\right|^r$. Hence, the summands of V_t^n are (on average) larger than the ones of $\mathsf{V}_t'^n$, consequently $\mathsf{V}_t^n - \mathsf{V}_t'^n > 0$ asymptotically and the test statistic explodes. Notice that the term $\Delta(n, i-1)$ enters in the definition of the α's in order to avoid a telescopic sum. Indeed, should this term be absent, the sum would be identically null for some sampling schemes, e.g., for those for which the sequence of the $\Delta(n,i)$'s is increasing.

We now derive the asymptotic distribution of $\mathsf{T}(X, \{t(n,i)\})$ under $\mathcal{H}_0$. In order to do so, we make the following additional assumptions.

$\mathcal{B}_1$ There is a sub-filtration $\{\mathcal{F}_t^0\}_{t\geq 0}$ of $\{\mathcal{F}_t\}_{t\geq 0}$ with the following properties:

(i) W, μ and σ are adapted to $\{\mathcal{F}_t^0\}_{t\geq 0}$;

(ii) any $\mathcal{F}_t^0$-martingale is also an $\mathcal{F}_t$-martingale;

(iii) each variable $t(n,i)$ is an $\mathcal{F}_t$-stopping time which, conditionally on $\mathcal{F}_{t(n,i-1)}$, is independent of the σ-field $\mathcal{F}^0 = \vee_{t\geq 0}\mathcal{F}_t^0$.

$\mathcal{B}_2$ For any $q > 0$ there exists a $\mathcal{F}_t^0$-optional positive process $\widetilde{a}(q)$, such that for all t, as $n \to \infty$,

$$r_n^{q-1} \sum_{i=1}^{N_t^n - 1} H_{t(n,i)} \Delta(n,i)^q \alpha(n,i) \xrightarrow{u.c.p.} \int_0^t H_s\, \widetilde{a}(q)_s\, ds, \tag{8}$$

for any càdlàg process H and where r_n is a diverging sequence of real numbers, $r_n \to \infty$.

Assumption $\mathcal{B}_1$ coincides with Assumption (C) of Hayashi et al. (2011). It implies that $\mathcal{H}_0$ holds. Assumption $\mathcal{B}_1$ holds, for example, when the $t(n,i)$'s are non-random and when the $t(n,i)$'s are independent from the processes (X, W, μ, σ). Assumption $\mathcal{B}_2$ holds for a number of random sampling schemes. For instance, it is satisfied when the $\Delta(n,i)$'s are iid positive random variables with finite moments. In this case, the convergence follows from the strong law of large numbers, the quantity $\tilde{a}(q)$ being a constant equal to the conditional expectation of $\Delta(n,i)^{q-1}$. We finally remark that, if the scheme $\{t(n,i)\}$ is such that $\forall i$, $\Delta(n,i) < \Delta(n,i+1)$ almost surely, then we have that $\alpha(n,i) = 0$ $\forall i \geq 2$ and, therefore, $\mathsf{T}(X, \{t(n,i)\})$ converges in distribution to a degenerate limit. To provide a solution to this problem, we discuss a modification of $\mathsf{T}(X, \{t(n,i)\})$ in Remark 1.

The following theorem establishes the limiting behaviour of $\mathsf{T}(X, \{t(n,i)\})$.

Theorem 3.1. *Let Assumptions $\mathcal{A}_1$, $\mathcal{A}_2$, $\mathcal{A}_3$, $\mathcal{B}_1$ and $\mathcal{B}_2$ hold. As $n \to \infty$ we have that*

$$\mathsf{T}(X, \{t(n,i)\}) \xrightarrow{\mathcal{F}^0-stably} \mathcal{N}(0,1),$$

where the convergence is $\mathcal{F}^0$-stable in law.

Proof. See Appendix A. □

Theorem 3.1 implies that the critical region for $\mathcal{H}_0$ ought to be of the form $\left\{|\mathsf{T}(X,\{t(n,i)\})| \geq q_{\alpha/2}\right\}$, where $q_{\alpha/2}$ denotes $\alpha/2$-quantile of standard normal distribution. This test is consistent for the hitting time alternative, defined by Assumption $\mathcal{H}_1$, under which we verify (via numerical simulations) that $\mathsf{T}(X,\{t(n,i)\}) \xrightarrow{p} \infty$.

Remark 1. *As we have emphasized above, the test may be in distribution asymptotically degenerate when $\alpha(n,i) = 0$ for all i. In this case the modified statistic*

$$\widetilde{\mathsf{T}}(X,\{t(n,i)\}) = \frac{\sum_{i=1}^{M_t^n} \Delta(n,2i-1)\left(\Delta(n,2i-1)^{-\frac{r}{2}}\left|\Delta_{2i-1}^n X\right|^r - \Delta(n,2i)^{-\frac{r}{2}}\left|\Delta_{2i}^n X\right|^r\right)\mathbf{1}_{\{\Delta(n,2i-1)>\Delta(n,2i)\}}}{\sqrt{\frac{2(\mu_{2r}-\mu_r^2)}{\mu_{2r}}\sum_{i=1}^{M_t^n}\Delta(n,2i-1)^{2-r}\left|\Delta_{2i-1}^n X\right|^{2r}\mathbf{1}_{\{\Delta(n,2i-1)>\Delta(n,2i)\}}}},$$

where $M_t^n = \left\lfloor \frac{N_t^n}{2} \right\rfloor$ and $r > 0$, provides a solution to the problem. The stable convergence of $\widetilde{\mathsf{T}}(X,\{t(n,i)\})$ requires that, for any $q > 0$, there exists a $\mathcal{F}_t^0$-optional positive process $\widetilde{a}'(q)$, such that for all t, as $n \to \infty$,

$$r_n^{q-1}\sum_{i=1}^{M_t^n} H_{t(n,i)}\Delta(n,i)^q \mathbf{1}_{\{\Delta(n,2i-1)>\Delta(n,2i)\}} \xrightarrow{u.c.p.} \int_0^t H_s\, \widetilde{a}'(q)_s\, ds,$$

for any càdlàg process H. Under this assumption, $\widetilde{\mathsf{T}}(X,\{t(n,i)\})$ converges $\mathcal{F}^0$-stably in law to $\mathcal{N}(0,1)$; the proof is analogous to the proof of Theorem 3.1 and omitted for the sake of brevity. It is worthwhile to point out that number of comparisons of the consecutive increments of X needed to compute $\widetilde{\mathsf{T}}(X,\{t(n,i)\})$ is smaller with respect to those necessary to compute $\mathsf{T}(X,\{t(n,i)\})$. Therefore, for small sample, it is preferable to use $\mathsf{T}(X,\{t(n,i)\})$, in order to have a more stable statistics.

3.2 Robustness to jumps

We now extend our testing theory for the case with jumps. For this purpose, we consider a more general version of Assumption $\mathcal{A}_1$. In particular, we assume that the efficient logarithmic price process is given by

$$Z_t = X_t + J_t, \tag{9}$$

where X_t is the process defined in Equation (3) and $\{J_t; t \geq 0\}$ denotes a finite activity jump process. The robustness to jumps can be achieved by computing our test statistic with power variation robustified using standard techniques: either by an appropriate choice of the power r, which guarantees that the Brownian term dominates the jump component (Barndorff-Nielsen et al., 2006) or by using truncation (Mancini, 2009). In the latter approach the robustified test statistic is defined as:

$$\widehat{\mathsf{T}}\left(Z,\{t(n,i)\}\right)=\frac{\sum_{i=1}^{N_t^n-1}\Delta(n,i)\left(\Delta(n,i)^{-\frac{r}{2}}\left|\Delta_i^n Z\right|^r-\Delta(n,i+1)^{-\frac{r}{2}}\left|\Delta_{i+1}^n Z\right|^r\right)\widehat{\alpha}(n,i)}{\sqrt{\frac{2(\mu_{2r}-\mu_r^2)}{\mu_{2r}}\sum_{i=1}^{N_t^n-1}\Delta(n,i)^{2-r}\left|\Delta_i^n Z\right|^{2r}\widehat{\alpha}(n,i)}},$$

where

$$\widehat{\alpha}(n,i)=\alpha(n,i)\mathbf{1}_{\left\{\left|\Delta_i^n Z\right|\le\vartheta(n,i)\right\}}\mathbf{1}_{\left\{\left|\Delta_{i+1}^n Z\right|\le\vartheta(n,i+1)\right\}},$$

and $\vartheta(n,i)$ denotes a sequence of threshold functions satisfying

$$\vartheta(n,i)\to 0,\quad\text{and}\quad\frac{\vartheta(n,i)}{\sqrt{\Delta(n,i)\log\Delta(n,i)}}\to\infty,$$

as $n\to\infty\,\forall\, i$, a.s.

Theorem 3.2. *Let Assumptions $\mathcal{A}_1$, $\mathcal{A}_2$, $\mathcal{A}_3$, $\mathcal{B}_1$ and $\mathcal{B}_2$ hold. Let Z be the process defined in equation* (9)*. As $n\to\infty$ we have that:*

- *if* $0<r<1$,

$$\mathsf{T}\left(Z,\{t(n,i)\}\right)\xrightarrow{\mathcal{F}^0-stably}\mathcal{N}\left(0,1\right);$$

- *if* $r>0$,

$$\widehat{\mathsf{T}}\left(Z,\{t(n,i)\}\right)\xrightarrow{\mathcal{F}^0-stably}\mathcal{N}\left(0,1\right).$$

Proof. See Appendix A. □

4 Endogenous sampling times and microstructure noise

In this section, we assume that Assumption $\mathcal{D}_1$ holds, that is a market microstructure noise contaminates the observations. Under the presence of a microstructure noise, the limiting behaviour of test statistic may depend on the probabilistic properties of the noise itself, the efficient price and the sampling times. Thus, additional assumptions on the different components are required to establish an asymptotic theory. To avoid putting strong constraints on the microstructure noise sequence, we consider a specific form for the sampling time's data generating process (DGP). Following Hayashi et al. (2011), we assume that the sampling scheme is a mixed renewal scheme, described in Assumption $\mathcal{C}_1$ below.

$\mathcal{C}_1$ The collection of random times $\{t\left(n,i\right)\}$ is generated by the recursive equation

$$t(n,i)=t(n,i-1)+\frac{1}{n}v_{t(n,i-1)}^n\varepsilon(n,i),\quad i=1,2\dots,\tag{10}$$

where $t(n,i)=0$, v^n is a sequence of positive $\mathcal{F}_t$-adapted processes, and $\{\varepsilon(n,i)\}$ is a double-sequence of iid random variables defined on $(\Omega,\mathcal{F},\mathbb{P})$ with finite fourth moment:

$$m_q = \mathbb{E}\left[\varepsilon(n,i)^q\right] < \infty, \quad q = 1,\ldots,4.$$

For each n, v^n is assumed to be a bounded semimartingale, and as $n\to\infty$, v^n converges in Skorokhod topology to a bounded semimartingale v.

The sampling scheme defined in equation (10) differs from the homologous of Hayashi et al. (2011) for a single peculiarity: the sequence v^n is required to be a bounded semimartingale and the first four moments of the $\varepsilon(n,i)$'s to be finite. The former assumption is needed for the standard estimates to the increments of v^n over infinitesimal time intervals. The latter to derive a CLT for the $\Delta(n,i)$'s.

Now, we put mild conditions on the microstructure noise sequence and the increments of the efficient price in, respectively, Assumptions $\mathcal{E}_1$ and $\mathcal{E}_2$ below.

$\mathcal{E}_1$ The sequence $\{U_{t(n,i)}\}$ has finite fourth moments and it is such that $\forall\, t>0$ and any càdlàg process H, as $n\to\infty$

$$\sum_{i=1}^{N_t^n-k_n} \Delta_i^n U\, H_{t(n,i-1)}\frac{1}{n} \xrightarrow{p} 0, \qquad \sum_{i=1}^{N_t^n-k_n} \left(\Delta_i^n U\right)^2 H_{t(n,i-1)}\frac{1}{n} \xrightarrow{p} \int_0^t u_s^2 H_s\, ds,$$

where u is an $\mathcal{F}_t^0$-optional stochastic process.

$\mathcal{E}_2$ There exist constants $C, C_\gamma > 0$, such that, for some $\alpha > \frac{1}{2}$ and any $\gamma > 0$, the following estimates hold

$$\left|\mathbb{E}\left[\Delta_i^n X \;\middle|\; \mathcal{F}_{t(n,i-1)} \vee \sigma\left(\Delta(n,i)\right)\right]\right| \leq C\,\Delta(n,i)^{\alpha}, \tag{11}$$

$$\mathbb{E}\left[\left|\Delta_i^n X\right|^{\gamma} \;\middle|\; \mathcal{F}_{t(n,i-1)} \vee \sigma\left(\Delta(n,i)\right)\right] \leq C_\gamma\,\Delta(n,i)^{\frac{\alpha\gamma}{2}}. \tag{12}$$

Assumption $\mathcal{E}_1$ incorporates standard models for the microstructure noise, for example, it allows $\left\{U_{t(n,i)}\right\}$ to be a q-dependent sequence. However, condition $\mathcal{E}_1$ holds for a more general class of models; for instance, it allows the sequence $\left\{U_{t(n,i)}\right\}$ to be non-stationary. Assumption $\mathcal{E}_2$ is required for testing the independence between sampling times and the noise. It provides sufficient conditions under which the increments of X are negligible with respect to the first differences of the noise sequence. If X is a semimartingale independent from the sampling times, Assumption $\mathcal{E}_2$ holds with $\alpha = 1$. In general, however, it does not require X to be a semimartingale and allows for a certain degree of dependence between $\Delta_i^n X$'s and $\Delta(n,i)$'s. The restriction $\alpha > \frac{1}{2}$ guarantees that the correlation between the $\Delta_i^n X$'s and the $\Delta(n,i)$'s does not affect the distribution of the test statistic proposed below. Finally, Assumption $\mathcal{E}_2$ allows the sampling times to be successive hitting times of a symmetric spatial grid by X.

4.1 Testing the independence between $\{U_{t(n,i)}\}$ and $\{t(n,i)\}$

Assumption $\mathcal{C}_1$ implies that:

$$\Delta(n,i) = \frac{1}{n} v^n_{t(n,i-1)} m_1 + \widetilde{\Delta}_i, \tag{13}$$

where $\widetilde{\Delta}_i = \Delta(n,i) - \mathbb{E}\left[\Delta(n,i) \mid \mathcal{F}_{i-1}\right]$ is a martingale-difference sequence. Thus, for testing the independence between $\left\{U_{t(n,i)}\right\}$ and $\{t\,(n,i)\}$ it is sufficient to test the statistical significance of the coefficient β_1 in the semi-parametric regression:

$$\Delta(n,i) = \beta_0(n,i) + \beta_1 \Delta^n_i U + \widetilde{\Delta}_i, \quad i = 1, 2, \ldots, N^n_t, \tag{14}$$

where $\beta_0(n,i) = \frac{1}{n} v^n_{t(n,i-1)} m_1$ is a time-varying intercept parameter. Statistical inference on β_1 is tangled by the fact that the $\Delta^n_i U$'s cannot be directly observed. However, if the efficient price process X verifies Assumption $\mathcal{E}_2$, the noise asymptotically dominates the efficient component of the observed price process. Thus, we can substitute unobserved $\Delta^n_i U$'s with $\Delta^n_i Y$'s in the regression (14). We address this problem analogously to time-varying regression models extensively studied in the literature (Zhang and Wu, 2012; Kalli and Griffin, 2014; Vogt, 2015; Zhang and Wu, 2015). However, we have a specific complication: the coefficient $\beta_0(n,i)$ is given by a realization of a semimartingale and, therefore, it is not a differentiable function of time as commonly assumed. Nonetheless, in the proof of Theorem 4.1, we show that standard results on the inference of $\beta_0(n,i)$ and β_1 continue to hold in our framework. In particular, we obtain a $\sqrt{n}$-consistent estimate of the parametric component β_1.

Formally, let k_n be a diverging sequence of integers such that $k_n/n \to 0$. Then an estimator of the regression coefficient β_1 is defined as:

$$\widehat{\beta}_1 = \frac{\sum_{i=1}^{N^n_t - k_n} \left(\Delta^n_i Y - \frac{1}{k_n} \sum_{j=0}^{k_n - 1} \Delta^n_j Y \right) \left(\Delta(n,i) - \frac{1}{k_n} \sum_{j=0}^{k_n-1} \Delta(n, i+j) \right)}{\sum_{i=1}^{N^n_t - k_n} \left(\Delta^n_i Y - \frac{1}{k_n} \sum_{j=0}^{k_n-1} \Delta^n_j Y \right)^2}.$$

This estimator corresponds to that of Gao and Hawthorne (2006) provided that an indicator kernel is used. Under the null $\beta_1 = 0$, de-trending the regressors $\Delta^n_i Y$'s is not necessary and $\widehat{\beta}_1$ can thus be replaced by

$$\widetilde{\beta}_1 = \frac{\sum_{i=1}^{N^n_t - k_n} \Delta^n_i Y \left(\Delta(n,i) - \frac{1}{k_n} \sum_{j=0}^{k_n-1} \Delta(n, i+j) \right)}{\sum_{i=1}^{N^n_t - k_n} \Delta^n_i Y^2}, \tag{15}$$

where $\frac{1}{k_n} \sum_{j=0}^{k_n-1} \Delta(n, i+j) \approx \beta_0(n,i)$ estimates the time-varying intercept. To test $\mathcal{H}_{U \perp \Delta}$ we propose to use the associated t-statistic, which takes the following form:

$$\mathsf{B}\left(Y,\{t(n,i)\}\right)=\frac{\sum_{i=1}^{N_t^n-k_n}\Delta_i Y\left(\Delta(n,i)-\frac{1}{k_n}\sum_{j=0}^{k_n-1}\Delta(n,i+j)\right)}{\sqrt{\sum_{i=1}^{N_t^n-k_n}\left(\Delta_i^n Y\left(\Delta(n,i)-\frac{1}{k_n}\sum_{j=0}^{k_n-1}\Delta(n,i+j)\right)\right)^2}}. \tag{16}$$

The asymptotic distribution of the test statistic $\mathsf{B}\left(Y,\{t(n,i)\}\right)$ is derived in the theorem below.

Theorem 4.1. *Let Assumptions $\mathcal{C}_1$, $\mathcal{D}_1$, $\mathcal{E}_1$ and $\mathcal{E}_2$ hold and $k_n=\lfloor\sqrt{n}\rfloor$. Under $\mathcal{H}_{U\perp\Delta}$, as $n\to\infty$ we have that*

$$\mathsf{B}\left(Y,\{t(n,i)\}\right)\overset{weakly}{\longrightarrow}\mathcal{N}\left(0,1\right). \tag{17}$$

Proof. See Appendix A. □

Theorem 4.1 shows that $\mathcal{H}_{U\perp\Delta}$ can be tested through an usual standard normal t-test. Hence, $\mathcal{H}_{U\perp\Delta}$ ought to be rejected for any sampling schemes which imply nonzero correlations between $\Delta(n,i)$'s and $\Delta_i^n U$'s. Indeed, the expression $\beta_0(n,i)+\beta_1\Delta_i^n U$ in the right hand-side of equation (14) is the local best linear predictor of $\Delta(n,i)$ given $\Delta_i^n U$. Hence, even if the relationship between $\Delta(n,i)$'s and $\Delta_i^n U$'s is not linear, non-zero correlation among them implies that the coefficient β_1 is non-zero.

4.2 Testing the independence of X and $\{t(n,i)\}$

In this section we propose a test for $\mathcal{H}_{X\perp\Delta}$, i.e. for the independence between the sampling scheme $\{t\left(n,i\right)\}$ and the efficient price process X. For this purpose, we combine the regression-based approach developed in the previous section with the pre-averaging technique, which allows to wash out, asymptotically, the microstructure noise. For simplicity, in this section we assume that $\left\{U_{t(n,i)}\right\}$ is a double-sequence of iid zero-mean random variables with finite moments of all orders.

Split the data on M_n blocks of length $\ell_n=\vartheta\, n^{\frac{1}{2}+\delta}$, with $\vartheta>0$ and $\delta\in\left(\frac{1}{6},\frac{1}{2}\right)$ (more details on the choice of δ are provided below). Let $g:\mathbb{R}\to\mathbb{R}$ be a non-zero real-valued function which is continuous and piecewise C^1, vanishes outside of the open interval $(0,1)$, and has a piecewise Lipschitz derivative g'. For each block $i=1,...,M_n$ and for a generic process V, let

$$\overline{V}_i^n=\sum_{j=1}^{\ell_n}g\left(\frac{j}{\ell_n}\right)\Delta_{i+j}^n V=\sum_{j=1}^{\ell_n}g\left(\frac{j}{\ell_n}\right)\left(V_{t(n,i+j)}-V_{t(n,i+j-1)}\right), \tag{18}$$

denote the pre-averaged increments of the process. In what follows, for the sake of exposition, we will adopt the notation $g_j^n = g(j/\ell_n)$. Analogously, we define the pre-averaged time-durations as

$$\overline{\Delta}_i^n = \sum_{j=1}^{\ell_n} g_j^n \, \Delta(n, i+j) = \sum_{j=1}^{\ell_n} g_j^n \, \left(t\,(n, i+j) - t\,(n, i+j-1) \right). \tag{19}$$

Intuitively, the pre-averaging eliminates the impact of the noise in each block of observations. Indeed, we have $\overline{Y}_i^n = \overline{X}_i^n + \overline{U}_i^n$, where $\overline{X}_i^n = O_p\left((\ell_n/n)^{1/2} \right)$ and $\overline{U}_i^n = O_p\left(\ell_n^{-1/2} \right)$. Consequently, if $\delta < \frac{1}{2}$, $\overline{X}_i^n$ dominates $\overline{U}_i^n$ asymptotically, so we have $\overline{Y}_i^n \approx \overline{X}_i^n$, for large n. We do not consider the common choice $\delta = 0$ (which gives $\ell_n \sim \sqrt{n}$) since in this case the stochastic orders of $\overline{X}_i^n$ and $\overline{U}_i^n$ are the same. In that case the contribution of $\overline{U}_i^n$ could bias our test statistic if the sequence $\left\{ U_{t(n,i)} \right\}$ is correlated with the sampling times. We aim to construct a test applicable in general settings, so we do not assume either independence between the noise and the sampling times or between noise and X. One may relax the restriction $\delta < \frac{1}{2}$ at the cost of a strong assumption of non-endogeneity of the noise.

Following the logic of the previous section we consider the semi-parametric regression

$$\overline{\Delta}_i^n = \beta_0(n, i) + \beta_1 \overline{Y}_i^n + \widetilde{\Delta}_i^n, \tag{20}$$

where $\beta_0(n, i)$ is a time-varying intercept and $\widetilde{\Delta}_i^n$ is a regression error. Under $\mathcal{H}_{X \perp \Delta}$, the coefficient β_1 is zero. Hence, we again use a version of a simple t-test to examine the null. Note that instead of using $\overline{\Delta}_i^n$ as the dependent variable one may consider a different function of the sampling times. In general, the choice of the dependent variable should guarantee high correlation with $\overline{X}_i^n$ under the alternative. We specify it as the "pre-averaged duration" as $\overline{\Delta}_i^n$ is highly correlated with $\overline{X}_i^n$ under the asymmetric hitting time sampling, which we consider as the benchmark alternative.

Our proposed test statistic takes the following form:

$$\overline{\mathsf{B}}\left(Y, \{t(n,i)\}\right) = \frac{\sum\limits_{i=1}^{M_n} \overline{Y}_{(i-1)\ell_n+1}^n \left(\overline{\Delta}_{(i-1)\ell_n+1}^n - \frac{1}{k_n} \sum\limits_{j=0}^{k_n-1} \overline{\Delta}_{(i-1)\ell_n+1+j}^n \right)}{\sqrt{\sum\limits_{i=1}^{M_n} \left(\overline{Y}_{(i-1)\ell_n+1}^n \left(\overline{\Delta}_{(i-1)\ell_n+1}^n - \frac{1}{k_n} \sum\limits_{j=0}^{k_n-1} \overline{\Delta}_{(i-1)\ell_n+1+j}^n \right) \right)^2}}, \tag{21}$$

where k_n denotes a deterministic sequence of integers and $\frac{1}{k_n} \sum\limits_{j=0}^{k_n-1} \overline{\Delta}_{i+j}^n \approx \beta_0(n, i)$ estimates the time-varying intercept. The next theorem provides the asymptotic distribution of $\overline{\mathsf{B}}\left(Y, \{t(n,i)\}\right)$ under $\mathcal{H}_{X \perp \Delta}$.

Theorem 4.2. *Assume that X is an Itō semimartingale defined by equations* (3) *and* (4)*. Let Assumption $\mathcal{C}_1$ hold. Assume, further, that the process v^n in equation* (10) *of Assumption $\mathcal{C}_1$ is independent from X. Set $\ell_n = \vartheta\, n^{1/2+\delta}$, $\vartheta > 0$ and $\delta \in \left(\frac{1}{6}, \frac{1}{2}\right)$. Let the sequence of integers k_n defined in* (21) *be such that $k_n \sim n^{\nu}$, where $\nu > 0$ is such that $\delta + \nu > \frac{1}{2}$ and $\nu < \frac{1}{2}$. Then, under*

$\mathcal{H}_{X\perp\Delta}$, *as* $n \to \infty$,

$$\overline{\mathsf{B}}\left(Y,\{t(n,i)\}\right) \overset{weakly}{\longrightarrow} \mathcal{N}\left(0,1\right).$$

Proof. See Appendix A. □

Theorem 4.2 imposes additional restriction on δ and k_n: $\delta > 1/6$, $k_n \sim n^{\nu}$, $\delta + \nu > \frac{1}{2}$ and $\nu < \frac{1}{2}$. They are required to guarantee the negligibility of the error terms of the asymptotic approximations $\overline{Y}_i^n \approx \overline{X}_i^n$ and $\frac{1}{k_n}\sum_{j=0}^{k_n-1}\overline{\Delta}_{i+j}^n \approx \beta_0(n,i)$. The critical region for $\mathcal{H}_{X\perp\Delta}$ is of the form $\mathcal{C}^{\mathcal{H}_{X\perp\Delta}} = \left\{\left|\overline{\mathsf{B}}\left(Y,\{t(n,i)\}\right)\right| \geq q_{\alpha/2}\right\}$, where $q_{\alpha/2}$ denotes $\alpha/2$-quantile of standard normal distribution. As for the case of the $\mathsf{T}\left(X,\{t(n,i)\}\right)$ test we verify (via numerical simulations) that, under the alternative defined by Assumption $\mathcal{H}_1$, the pre-averaged based test delivers a unit power, i.e. $\overline{\mathsf{B}}\left(Y,\{t(n,i)\}\right) \overset{p}{\longrightarrow} \infty$.

5 Monte Carlo assessment of finite sample performances

In this section we study the finite sample performance of the proposed tests in different settings. We consider first the case of a noiseless price process to assess the power and the size of the test statistic in (6) and we compare them with those of the test proposed by Li et al. (2014), which is, to the best of our knowledge, the benchmark in the reference literature for this study. To do so, we consider a scenario in which (noiseless) prices are observed at random times, generated according to a suitable DGP which may (to asses the power of the test) or may not (to asses the size of the test) feature time-price dependence. Next, we provide information on the power and size of the test statistic B and $\overline{\mathsf{B}}$ defined, respectively, in (16) and (21).

In all the simulation settings, the efficient log-price X is generated according the one-factor stochastic volatility model

$$\begin{aligned} dX_t &= \mu\, dt + c_\sigma\, \sigma_t\, dW_{X,t}, \\ d\log\sigma_t^2 &= \left(\alpha - \beta\, \log\sigma_t^2\right)\, dt + \eta\, dW_{\sigma,t}, \end{aligned} \tag{22}$$

where $W_{\sigma,t}$ and $W_{X,t}$ are two Brownian motions with $\mathrm{corr}\left(dW_{\sigma,t}, dW_{X,t}\right) = \rho\, dt$. We adopt the values for the parameters α, β, η, μ and ρ as in Andersen et al. (2002) on S&P500. The volatility factor c_σ can be tuned to generate different scenarios. It will be equal to $c_\sigma = 4$ (which corresponds to a daily volatility of roughly 3%), unless otherwise specified.

5.1 Testing for endogeneity in absence of microstructure noise

In the first simulated experiment, we generate 10^4 sample paths of a process X observed on an interval $[0,1]$ under two different sampling schemes. Under the null, the sampling times $\{t\left(n,i\right)\}$

are obtained via the ACD model

$$\begin{cases} t\,(n,0) = 0, \\ t\,(n,i) = \psi\,(n,i)\; e\,(n,i)\,. \end{cases} \tag{23}$$

where $\psi\,(n,i) = \frac{1}{n} + 0.2\,t\,(n,i-1) + 0.6\,\psi\,(n,i-1)$ and $e\,(n,i) \sim \exp\,(1/2)$ is a sequence of iid exponentially distributed random variables with mean 1/2. We consider three values for the frequency parameter n, which are $n = 11700, 23400, 46800$. To illustrate, for $n = 46800$ we get, on average, roughly 28000 sampling times in the interval $[0, 1]$ that correspond to, approximately, 1.2 sampling times per second (assuming a trading day of 6.5 hours). Smaller value of n generates more idle scenarios, i.e. price paths characterized by a less intense trading activity. Accordingly, the integer n defines the liquidity status of the simulated path.

Under the alternative, the endogenous sampling times are defined recursively according to the scheme

$$\begin{cases} t(n,0) = 0, \\ t(n,i+1) = \inf\left\{t > t\,(n,i) \;\middle|\; X_t - X_{t(n,i)} > n^{1/2}a \vee X_t - X_{t(n,i)} < -n^{1/2}b\right\}, \quad \forall i > 0. \end{cases} \tag{24}$$

for some barriers $a, b > 0$. We consider, as for the null, the three liquidity scenarios characterized by $n = 11700, 23400, 46800$ and, in addition, we consider a range of values of the barriers a and b. We remark that, for both sampling schemes and each generated sample, the number of observations N_t^n, used to compute the test statistic, is random and equal to the number of instants $t(n,i)$'s falling in the interval $[0, 1]$.

Table 1 reports size and power of $\mathsf{T}\,(X, \{t\,(n,i)\})$ under the null and three different alternative: a symmetric barrier ($a = b = 0.02$), a slightly asymmetric barrier ($a = 0.03$ and $b = 0.02$) and a more pronounced asymmetric barrier ($a = 0.04$ and $b = 0.02$). For comparison we report, in Table 2, the same quantities for the test of Li et al. (2014) (denoted, henceforth, LMRZZ). The table shows that the $\mathsf{T}\,(X, \{t\,(n,i)\})$ test is correctly sized under exogenous sampling. Second, the test has unit power against the hitting time alternatives, for all the considered values of a and b. On the other side, the lack of power of the LMRZZ test in the alternative with symmetric barriers ($a = b = 0.02$) is explained by noticing that, despite the endogeneity of time, the realized tricity still converges to zero in this case. In the slightly asymmetric case, $\mathsf{T}\,(X, \{t\,(n,i)\})$ largely outperforms LMRZZ. Intuitively, the difference in performances of the two approaches can be explained by the fact that the $\mathsf{T}\,(X, \{t\,(n,i)\})$ test compares the sizes of the increments of the observed process corresponding to different $\Delta(n,i)$'s and not only the sizes of increments with different signs, which affects the limit of the realized tricity.

Table 1: Size and power of the $\mathsf{T}(X, \{t(n,i)\})$ test for endogeneity in absence of microstructure noise.

$(1-\alpha)(\%)$	n	Null	(a,b) $(0.02, 0.02)$	$(0.03, 0.02)$	$(0.04, 0.02)$
	11700	9.77	100.00	100.00	100.00
90.0	23400	10.03	100.00	100.00	100.00
	46800	10.20	100.00	100.00	100.00
	11700	4.86	100.00	100.00	100.00
95.0	23400	4.70	100.00	100.00	100.00
	46800	4.88	100.00	100.00	100.00
	11700	1.11	100.00	100.00	100.00
99.0	23400	0.99	100.00	100.00	100.00
	46800	0.99	100.00	100.00	100.00
	11700	0.16	100.00	100.00	100.00
99.9	23400	0.09	100.00	100.00	100.00
	46800	0.12	100.00	100.00	100.00

Note. The table reports the percentage of rejections of the double-sided test statistic $\mathsf{T}(X, \{t(n,i)\})$ when the sampling times are independent of X (null) and under three alternatives: a symmetric barrier ($a = b = 0.02$), a slightly asymmetric barrier ($a = 0.03$ and $b = 0.02$) and a more pronounced asymmetric barrier ($a = 0.04$ and $b = 0.02$). In all the simulated scenarios, the process X follows the stochastic volatility model defined by the equation (22).

5.2 Finite sample performances of the $\mathsf{B}(Y, \{t(n,i)\})$ test

In this simulated experiment, we generate sample paths of the process $Y_{t(n,i)} = X_{t(n,i)} + U_{t(n,i)}$ defined in equation (5). As for the other settings considered so far, the efficient price process $X_{t(n,i)}$ is simulated according to equation (22). The sampling instants $t(n,i)$'s are simulated according to the ACD model in equation (23). Two specifications of the microstructure noise U are considered. In the first we generate $U_{t(n,i)}$ as a sequence of iid random variables such that $U_{t(n,i)}$ is distributed according to $\mathcal{N}(0, \omega_0^2)$ and $\omega_0^2 = \xi \, \langle \sigma_t^2 \rangle$, where $\langle \sigma_t^2 \rangle$ is the average volatility of X (see equation (22) for details) and ξ is the noise-to-signal ratio at the highest frequency, whose (percentage) values are reported in Table 3. In this setting, the microstructure noise is independent from $\{t(n,i)\}$. We use the so obtained simulated paths to determine the finite sample size of the test $\mathsf{B}(Y, \{t(n,i)\})$.

In the second, which is designed to determine the power of the test, we induce a time-noise

Table 2: Size and power of the LMRZZ test for endogeneity in absence of microstructure noise.

$(1-\alpha)(\%)$	n	Null	(a,b)		
			$(0.02, 0.02)$	$(0.03, 0.02)$	$(0.04, 0.02)$
	11700	10.46	0.63	88.75	99.99
90.0	23400	10.50	0.59	99.70	100.00
	46800	9.86	0.59	100.00	100.00
	11700	5.19	0.10	75.09	99.90
95.0	23400	5.20	0.07	98.74	100.00
	46800	5.07	0.06	100.00	100.00
	11700	1.09	0.01	37.80	98.99
99.0	23400	1.11	0.00	89.35	100.00
	46800	1.12	0.00	99.98	100.00
	11700	0.13	0.00	7.30	88.68
99.9	23400	0.13	0.00	52.54	99.99
	46800	0.11	0.00	98.78	100.00

Note. The table reports the percentage of rejections of the double-sided test statistic LMRZZ when the sampling times are independent of X (null) and under three alternatives: a symmetric barrier ($a = b = 0.02$), a slightly asymmetric barrier ($a = 0.03$ and $b = 0.02$) and a more pronounced asymmetric barrier ($a = 0.04$ and $b = 0.02$). In all the simulated scenarios, the process X follows the stochastic volatility model defined by the equation (22).

dependence via the following mechanism:

$$U_{t(n,i)} \sim \begin{cases} \omega_1\left(\exp(3) - \exp(1)\right), & \text{if } \Delta(n,i) > \mathsf{med}(\Delta(n,1),\ldots,\Delta(n,N_t^n)), \\ \omega_1\left(\exp(1) - \exp(3)\right), & \text{if } \Delta(n,i) \leq \mathsf{med}(\Delta(n,1),\ldots,\Delta(n,N_t^n)), \end{cases} \tag{25}$$

where med denotes the median and ω_1 is chosen in such a way that $\mathbb{E}\left[U^2_{t(n,i)}\right] \approx \omega_0^2$.

Finally, to fully account for all the significant finite sample distortions at high-frequency, prices are (in all the simulated paths) rounded to one cent.

Table 3 summarizes the results of this simulated experiment by showing that the test is appropriately sized and delivers, under the alternative defined by equation (25), a unit power.

Table 3: Size and power of test statistic $\mathsf{B}(Y, \{t(n,i)\})$.

$(1-\alpha)(\%)$	n	Size			Power		
		$\xi(\%)$			$\xi(\%)$		
		0.10	0.50	1.00	0.10	0.50	1.00
90.00	11700	8.78	8.32	8.32	100.00	100.00	100.00
	23400	8.42	8.62	8.08	100.00	100.00	100.00
	46800	8.04	7.86	7.12	100.00	100.00	100.00
95.00	11700	4.26	3.86	3.84	100.00	100.00	100.00
	23400	4.08	3.96	3.62	100.00	100.00	100.00
	46800	4.00	3.82	3.28	100.00	100.00	100.00
99.00	11700	0.56	0.98	0.80	100.00	100.00	100.00
	23400	0.62	0.46	0.44	100.00	100.00	100.00
	46800	0.66	0.72	0.44	100.00	100.00	100.00
99.90	11700	0.06	0.12	0.12	100.00	100.00	100.00
	23400	0.06	0.06	0.06	100.00	100.00	100.00
	46800	0.04	0.04	0.00	100.00	100.00	100.00

Note. The table reports rejections rates, for different significance levels α (first column) of the test statistic $\mathsf{B}(Y, \{t(n,i)\})$ defined in equation (16). We report the rates under the null of no dependence between the noise and the sampling times (size of the test) and under the alternative defined by equation (25) (power of the test). The parameter ξ indicates the (percentage) noise-to-signal ratio at the highest frequency. Sampling times are generated, both under the null and the alternative, according to the ACD model in equation (23), with the corresponding value of n reported in the second column. Only sampling times that fall within the interval $[0,1]$ are considered, generating a random number of them equal to $N_1^n = \inf(i : t(n,i) > 1) - 1$.

5.3 Finite sample performances of the $\overline{\mathsf{B}}(Y, \{t(n,i)\})$ test

To assess the finite sample size and power of the test $\overline{\mathsf{B}}(Y, \{t(n,i)\})$, defined in equation (21) (and to compare them with their LMRZZ test counterparts), we generate paths of the noise-contaminated price process $Y_{t(n,i)} = X_{t(n,i)} + U_{t(n,i)}$ with the sampling scheme $\{t(n,i)\}$ as in equation (23), the process $X_{t(n,i)}$ as in the stochastic differential equation (22) and the noise $U_{t(n,i)}$ as in (25). We thus allow, under the null, a dependence between the noise and the sampling times, while keeping the efficient price X independent from them (as it is prescribed under the null). This modeling choice is made specifically to test the robustness of the pre-average method to noise-sampling time dependence and it is also justified by the empirical findings described in

Section 6.

Finally, we determine the power of the test by changing the data generating process of the $\{t\,(n,i)\}$ to the hitting time sampling model in equation (24), while keeping all other details unchanged.

As for the other simulated scenarios considered so far, observed prices are (both under the null and the alternative) rounded to one cent.

Tables 4 and Table 5 illustrate the results of the experiment. The former reports the percentage rejection rates for $\overline{\mathsf{B}}\,(Y, \{t(n,i)\})$, while the latter shows the same quantities but for the LMRZZ test. The comparison is in net favour of $\overline{\mathsf{B}}\,(Y, \{t(n,i)\})$, which is more correctly sized and deliver a significantly higher power.

Table 4: Size and power of test statistic $\overline{\mathsf{B}}\,(Y, \{t(n,i)\})$.

		Size			Power		
		$\xi(\%)$			$\xi(\%)$		
$(1-\alpha)(\%)$	n	0.10	0.50	1.00	0.10	0.50	1.00
	11700	12.16	12.12	11.94	62.32	59.56	56.78
90.00	23400	11.64	11.36	11.30	74.44	71.74	68.36
	46800	10.76	10.44	10.32	86.42	84.02	81.02
	11700	6.42	6.28	5.86	49.48	46.58	43.46
95.00	23400	5.96	5.98	5.78	62.96	59.30	55.40
	46800	5.24	5.36	5.22	77.70	74.50	70.80
	11700	1.08	1.20	1.12	24.18	22.42	20.28
99.00	23400	1.22	1.12	1.22	36.64	33.16	30.22
	46800	1.12	1.08	1.22	55.34	51.82	47.38
	11700	0.12	0.12	0.14	6.74	6.00	4.96
99.90	23400	0.16	0.14	0.18	13.56	11.76	9.62
	46800	0.06	0.04	0.08	27.46	24.14	20.54

Note. The table reports rejections rates, for different significance levels α (first column) of the test statistic $\overline{\mathsf{B}}\,(Y, \{t(n,i)\})$, defined in equation (21). We report the rates under the null of no dependence between the efficient time process X and the sampling times $\{t\,(n,i)\}$ (size of the test) and under the alternative defined by the hitting time sampling scheme of equation (24) (power of the test). The parameter ξ indicates the (percentage) noise-to-signal ratio at the highest frequency. Noise and sampling times are generated as dependent random variables (both under the null and the alternative) using the mechanism described in equation (25). Sampling times are generated, under the null, according to the ACD model in equation (23), with the corresponding value of n reported in the second column, which also represents, under the alternative, the parameter n of the hitting sampling scheme (24). In both cases, only sampling times that fall within the interval $[0,1]$ are considered, generating a random number of them equal to $N_1^n = \inf\,(i\,:\,t(n,i) > 1) - 1$.

Table 5: Size and power of test statistic LMRZZ.

		Size			Power		
		$\xi(\%)$			$\xi(\%)$		
$(1-\alpha)(\%)$	n	0.10	0.50	1.00	0.10	0.50	1.00
	11700	3.02	1.56	1.16	10.36	4.04	3.20
90.00	23400	2.42	1.34	1.26	8.22	3.68	3.30
	46800	2.48	1.40	1.34	7.10	3.28	3.30
	11700	1.36	0.54	0.40	6.04	1.70	1.46
95.00	23400	1.10	0.46	0.28	4.62	1.86	1.42
	46800	1.12	0.64	0.52	3.50	1.54	1.56
	11700	0.16	0.00	0.00	1.62	0.50	0.34
99.00	23400	0.08	0.00	0.00	1.32	0.28	0.26
	46800	0.14	0.02	0.02	0.70	0.24	0.30
	11700	0.00	0.00	0.00	0.16	0.08	0.02
99.90	23400	0.02	0.00	0.00	0.14	0.02	0.02
	46800	0.00	0.00	0.00	0.08	0.02	0.02

Note. The table reports rejections rates, for different significance levels α (first column) of the test statistic LMRZZ. We report the rates under the null of no dependence between the efficient time process X and the sampling times $\{t\,(n,i)\}$ (size of the test) and under the alternative defined by the hitting time sampling scheme of equation (24) (power of the test). The parameter ξ indicates the (percentage) noise-to-signal ratio at the highest frequency. Noise and sampling times are generated as dependent random variables (both under the null and the alternative) using the mechanism described in equation (25). Sampling times are generated, under the null, according to the ACD model in equation (23), with the corresponding value of n reported in the second column, which also represents, under the alternative, the parameter n of the hitting sampling scheme (24). In both cases, only sampling times that fall within the interval $[0,1]$ are considered, generating a random number of sampling times equal to $N_1^n = \inf\,(i\,:\,t(n,i) > 1) - 1$.

6 Time endogeneity in financial assets' data

We use our newly derived tests as a microscope of the structure of financial asset prices to uncover a dependence between the components of the observed price (i.e. the efficient price and the microstructure noise) and the instants at which it is sampled.

First, we investigate time endogeneity at the highest possible frequency. To do so, we take all the transactions of the top 250 liquid (in terms of total traded volume) stocks of the NYSE in 2014. For each day and for each stock, we compute, whenever at least 1000 transactions are available, the noise-time dependence test $\mathsf{B}\left(Y, \{t(n,i)\}\right)$ and the efficient price-noise dependence test $\overline{\mathsf{B}}\left(Y, \{t(n,i)\}\right)$, using all the observed prices and the corresponding trading times (tick-by-tick sampling scheme). On average, a single test is obtained using 3251.54 observations, which is the average number of transactions per stock per day. In order to provide solid empirical evidence, we pool the results of the two tests across days and stocks.

Figure 1 summarizes our empirical findings. We report, respectively on the left and right panel, the histograms of the pooled samples of the tests $\mathsf{B}\left(Y, \{t(n,i)\}\right)$ and $\overline{\mathsf{B}}\left(Y, \{t(n,i)\}\right)$. As a red dotted line, we superimpose the probability density function of a standard Gaussian variable. This empirical application highlights two crucial features of asset price dynamics that have never been documented so far to the best of our knowledge. First, the efficient price process does not show a dependence on trading instants, not at least one that our tests is capable to detect (as, for example, that of a hitting time barrier). Second, the microstructure noise and the trading time cannot be assumed to be independent, as there is a clear violation of the null documented by the $\mathsf{B}\left(Y, \{t(n,i)\}\right)$ test statistic . This result points toward an extension of some important studies of the literature (see Jacod et al., 2017; Merrick and Linton, 2022, among others) for inference regarding the join distribution of noise and sampling times.

As a further empirical exercize, using the same dataset and the same pooling strategy, we consider other two random sampling schemes. The first, which will be addressed, following Oomen (2006), as business time sampling, consists in sampling the price process every time that the number of traded prices is a multiple of a given integer quantity. So, for example, prices are sampled every ten, twenty etc. transactions. The tick-by-tick sampling is therefore a particular case of business time sampling. Trivially, the total number of sampling instants is a random quantity. In the second, which we address as dollar-volume sampling, we follow the idea of volume bucketing described in Easley et al. (2012). This scheme consists in sampling a traded price every time that a given amount (a bucket) of dollar-volume is traded. The bucket is chosen as a percentage of the total volume traded during the day. The larger the bucket the larger the average distance between two consecutive sampling points (and, accordingly, the smaller the average sampling frequency). So, for example, if prices are sampled every time that an eightieth of the total daily dollar-volume is traded we will have a sampling scheme with exactly 80 points. Nevertheless, given the discrete nature of the trading process, the sampling scheme may generate repeated times. Being repetitions

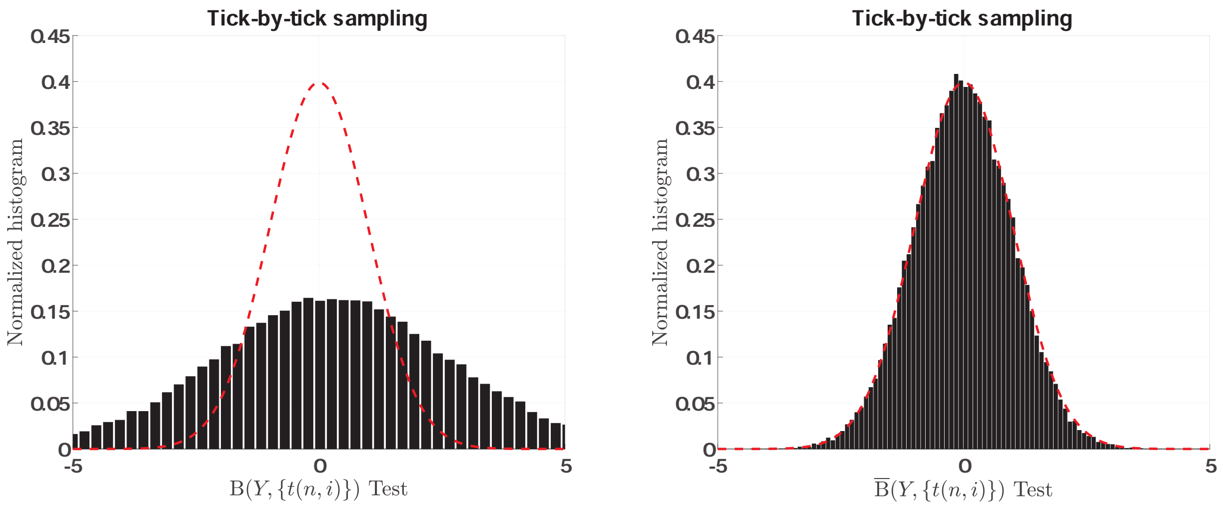

Figure 1: We report the (normalized) histograms of the samples obtained by pooling, across days and stocks, the values of the test statistic $\mathsf{B}(Y, \{t(n,i)\})$ (left panel) and $\overline{\mathsf{B}}(Y, \{t(n,i)\})$ (right panel). To obtain the two samples, we use all the transactions of the top 250 liquid (in terms of average traded volume) stocks of the NYSE in 2014. We compute the tests for each stock in the sample and for days with at least 1000 transactions. Accordingly, tests are computed on a daily basis using all the time stamps (tick-by-tick sampling) and the corresponding transaction prices. Superimposed, as a red dotted line, we report the probability density function of a standard Gaussian variable.

not allowed, they are removed producing, as a consequence, a random number of sampling points, as for the business time sampling scheme.

Table 6 summarizes the results of this empirical investigation. We report, for different significance level, the rejection rates of the tests (computed, as mentioned above, by pooling across days and stocks) $\mathsf{T}(X, \{t(n,i)\})$, $\mathsf{B}(Y, \{t(n,i)\})$ and $\overline{\mathsf{B}}(Y, \{t(n,i)\})$ defined, respectively, in equations (6), (16) and (21). For each sampling scheme we also indicates with $\overline{\Delta}$ the average time distance (in minutes) between two consecutive points of the sampling partition. We consider different average sampling frequencies, from high ($\overline{\Delta} \approx 0.20$ minutes) to low ($\overline{\Delta} \approx 11$ minutes). For the case of the test $\mathsf{T}(X, \{t(n,i)\})$, we report rejection rates only if $\overline{\Delta} \geq 7$ minutes. We do so because the test is reliable if the microstructure noise is negligible. The figures in Table 6 confirm, for the cases of $\mathsf{B}(Y, \{t(n,i)\})$ and $\overline{\mathsf{B}}(Y, \{t(n,i)\})$, the same findings obtained with the tick-by-tick sampling: the efficient price is independent from both sampling schemes, while the microstructure noise shows a strong level of dependence. We remark that these two tests are designed to work for high-frequency data, so the rejection rates for large value of $\overline{\Delta} \approx 11$ should not be considered as indicative. Finally, the rejection rates associated to $\mathsf{T}(X, \{t(n,i)\})$ reveal that both the business time and the dollar volume sampling scheme, at low frequency, are endogenous.

Table 6: Empirical rejection rates under business time sampling and dollar-volume time sampling.

	Business time sampling				Dollar-volume time sampling			
$(1-\alpha)(\%)$	$\overline{\Delta}$	T	B	$\overline{\mathsf{B}}$	$\overline{\Delta}$	T	B	$\overline{\mathsf{B}}$
99.00	0.19	-	29.70	0.98	0.56	-	17.88	0.58
	1.56	-	9.04	0.39	1.47	-	7.55	0.33
	5.46	-	1.64	0.07	3.72	-	2.04	0.07
	7.03	14.49	1.20	0.06	7.42	14.25	0.70	0.00
	11.00	12.12	0.53	0.01	11.01	12.16	0.40	0.00
95.00	0.19	-	42.34	4.78	0.56	-	30.89	4.03
	1.56	-	19.67	3.81	1.47	-	18.07	3.53
	5.46	-	7.64	3.12	3.72	-	8.58	2.82
	7.03	23.57	6.28	2.73	7.42	21.94	5.34	1.97
	11.00	19.73	4.74	1.70	11.01	18.82	4.34	0.49
90.00	0.19	-	50.10	9.76	0.56	-	39.39	8.78
	1.56	-	27.75	8.82	1.47	-	26.26	8.30
	5.46	-	14.31	9.15	3.72	-	15.41	7.96
	7.03	30.06	12.63	9.03	7.42	27.63	11.36	7.61
	11.00	25.32	10.84	8.34	11.01	23.83	10.29	7.58

Note. The table reports empirical rejection rates, for different significance level α (first column), of the tests $\mathsf{T}(X, \{t(n,i)\})$, $\mathsf{B}(Y, \{t(n,i)\})$ and $\overline{\mathsf{B}}(Y, \{t(n,i)\})$ defined, respectively, in equations (6), (16) and (21). Data are all transactions (and the relative timestamps) of the top 250 liquid (in terms of total traded volume) NYSE stocks in the year 2014. Data are sampled, as indicated in the first row of the table, either according to the business time or the dollar-volume schemes, both described in the main text. For each type of sampling scheme we report, in the column indicated as $\overline{\Delta}$, the average (across all days and all stocks) time distance in minutes between two consecutive points of the scheme. In the case of the test T, we report only rejections rates relative to a $\overline{\Delta} \geq 7$ minutes, since the test is designed to work in absence of microstructural noise.

7 Conclusions

Trading occurs at random instants. The non-deterministic nature of the sampling process raises the question of its endogeneity. The implications are indeed innumerable: both form a purely econometric point of view but also for a better understanding of the price formation mechanisms.

This study provides new tools to shed light on time endogeneity in financial markets. We derive test statistics for detecting price-time dependence in the presence of microstructure noise. Our tests are designed in such a way that we can separately identify the dependence between sampling times and the microstructure noise and between sampling times and the efficient component of the observed price process.

When applied on stocks' data, we document that microstructure noise and trade arrival instants cannot be considered as independent variables. On the other side, we provide statistically robust evidence that the efficient component of the observed prices does not show a dependence with the trading instants of the kind defined by symmetric or asymmetric hitting time sampling.

Finally, using business time and dollar-volume sampling schemes with moderate (average) sampling frequencies (ones for which the impact of the microstructure noise is negligible) we document that the observed price process (now assumed noiseless) depends on the corresponding sampling times.

References

Andersen, T., L. Benzoni, and J. Lund (2002). An empirical investigation of continuous-time equity return models. *Journal of Finance 57*, 1239–1284.

Barndorff-Nielsen, O. E., S. E. Graversen, J. Jacod, M. Podolskij, and N. Shephard (2006). A central limit theorem for realised power and bipower variations of continuous semimartingales. In *From stochastic calculus to mathematical finance*, pp. 33–68. Springer.

Barndorff-Nielsen, O. E. and N. Shephard (2002). Econometric analysis of realized volatility and its use in estimating stochastic volatility models. *Journal of the Royal Statistical Society: Series B (Statistical Methodology) 64*(2), 253–280.

Barndorff-Nielsen, O. E. and N. Shephard (2006). Power variation and time change. *Theory of Probability & Its Applications 50*(1), 1–15.

Barndorff-Nielsen, O. E., N. Shephard, and M. Winkel (2006). Limit theorems for multipower variation in the presence of jumps. *Stochastic Processes and Their Applications 116*(5), 796–806.

Bibinger, M. and P. A. Mykland (2016). Inference for multi-dimensional high-frequency data with an application to conditional independence testing. *Scandinavian Journal of Statistics 43*(4), 1078–1102.

Christensen, K., S. Kinnebrock, and M. Podolskij (2010). Pre-averaging estimators of the ex-post covariance matrix in noisy diffusion models with non-synchronous data. *Journal of Econometrics 159*(1), 116–133.

Cui, W. (2021). Laplace estimator of integrated volatility when sampling times are endogenous. *Journal of Business & Economic Statistics 0*(0), 1–13.

Dimitriadis, T. and R. Halbleib (2022). Realized quantiles*. *Journal of Business & Economic Statistics 40*(3), 1346–1361.

Easley, D., M. M. López de Prado, and M. O'Hara (2012, 03). Flow Toxicity and Liquidity in a High-frequency World. *The Review of Financial Studies 25*(5), 1457–1493.

Fukasawa, M. (2010a). Central limit theorem for the realized volatility based on tick time sampling. *Finance and Stochastics 14*(2), 209–233.

Fukasawa, M. (2010b). Realized volatility with stochastic sampling. *Stochastic Processes and Their Applications 120*(6), 829–852.

Fukasawa, M. and M. Rosenbaum (2012). Central limit theorems for realized volatility under hitting times of an irregular grid. *Stochastic processes and Their Applications 122*(12), 3901–3920.

Gao, J. and K. Hawthorne (2006). Semiparametric estimation and testing of the trend of temperature series. *The Econometrics Journal 9*(2), 332–355.

Hayashi, T., J. Jacod, and N. Yoshida (2011). Irregular sampling and central limit theorems for power variations: The continuous case. *Annales de l'IHP Probabilités et statistiques 47*(4), 1197–1218.

Jacod, J. (2008). Asymptotic properties of realized power variations and related functionals of semimartingales. *Stochastic Processes and Their Applications 118*(4), 517–559.

Jacod, J. (2012). Statistics and high frequency data. *Unpublished manuscript*.

Jacod, J., Y. Li, P. A. Mykland, M. Podolskij, and M. Vetter (2009). Microstructure noise in the continuous case: The pre-averaging approach. *Stochastic Processes and their Applications 119*(7), 2249–2276.

Jacod, J., Y. Li, and X. Zheng (2017). Statistical properties of microstructure noise. *Econometrica 85*(4), 1133–1174.

Jacod, J., M. Podolskij, and M. Vetter (2010). Limit theorems for moving averages of discretized processes plus noise. *The Annals of Statistics 38*(3), 1478 – 1545.

Jacod, J. and P. Protter (2004). *Probability essentials*. Springer Science & Business Media.

Kalli, M. and J. E. Griffin (2014). Time-varying sparsity in dynamic regression models. *Journal of Econometrics 178*(2), 779–793.

Li, Y., P. A. Mykland, E. Renault, L. Zhang, and X. Zheng (2014). Realized volatility when sampling times are possibly endogenous. *Econometric Theory 30*(3), 580–605.

Li, Y., Z. Zhang, and X. Zheng (2013). Volatility inference in the presence of both endogenous time and microstructure noise. *Stochastic Processes and their Applications 123*(7), 2696–2727.

Mancini, C. (2009). Non-parametric threshold estimation for models with stochastic diffusion coefficient and jumps. *Scandinavian Journal of Statistics 36*(2), 270–296.

Merrick, Z. and O. Linton (2022). A ReMeDI for microstructure noise. *Econometrica 90*(1), 367–389.

Mykland, P. A. and L. Zhang (2006). Anova for diffusions and ito processes. *The Annals of Statistics 34*(4), 1931–1963.

Oomen, R. C. A. (2006). Properties of realized variance under alternative sampling schemes. *Journal of Business & Economic Statistics 24*(2), 219–237.

Podolskij, M. and M. Vetter (2009a). Bipower-type estimation in a noisy diffusion setting. *Stochastic Processes and their Applications 119*(9), 2803–2831.

Podolskij, M. and M. Vetter (2009b). Estimation of volatility functionals in the simultaneous presence of microstructure noise and jumps. *Bernoulli 15*(3), 634 – 658.

Podolskij, M. and M. Vetter (2010). Understanding limit theorems for semimartingales: a short survey. *Statistica Neerlandica 64*(3), 329–351.

Potiron, Y. and P. A. Mykland (2017). Estimation of integrated quadratic covariation with endogenous sampling times. *Journal of Econometrics 197*(1), 20–41.

Vogt, M. (2015). Testing for structural change in time-varying nonparametric regression models. *Econometric Theory 31*(4), 811–859.

Zhang, T. and W. B. Wu (2012). Inference of time-varying regression models. *The Annals of Statistics 40*(3), 1376 – 1402.

Zhang, T. and W. B. Wu (2015). Time-varying nonlinear regression models: Nonparametric estimation and model selection. *The Annals of Statistics 43*(2), 741 – 768.

A Appendix: Proofs

A.1 Proofs of results in absence of microstructural noise

Proof of Theorem 3.1 .

Let $\beta_i^n = \sigma_{t(n,i-1)} \Delta(n,i)^{-1/2} \Delta_i^n W$ and $\beta_i'^n = \sigma_{t(n,i-2)} \Delta(n,i)^{-1/2} \Delta_i^n W$ denote the two approximations for the rescaled increments of X and consider an approximation for the numerator of $\mathsf{T}(X, \{t(n,i)\})$ defined by

$$\mathsf{U}_t^n = \sqrt{r_n} \sum_{i=1}^{N_t^n - 1} \Delta(n,i) \left(|\beta_i^n|^r - \left|\beta_{i+1}'^n\right|^r \right) \alpha(n,i).$$

The proof consists of showing three convergence results:

1. A central limit theorem for the approximation:
$$\mathsf{U}_t^n \xrightarrow{\mathcal{F}^0-\text{stably}} \sqrt{2\left(\mu_{2r} - \mu_r^2\right)} \int_0^t \sigma_s^r \sqrt{\widetilde{a}(2)_s}\, dW_s', \tag{26}$$
where W' is a Winer process independent from W.
2. The asymptotic negligibility of the approximation error:
$$\mathsf{U}_t^n - \sqrt{r_n} \sum_{i=1}^{N_t^n - 1} \Delta(n,i) \left(\Delta(n,i)^{-\frac{r}{2}} |\Delta_i^n X|^r - \Delta(n,i+1)^{-\frac{r}{2}} \left|\Delta_{i+1}^n X\right|^r \right) \alpha(n,i) \xrightarrow{u.c.p.} 0. \tag{27}$$
3. The convergence of the denominator of the test statistic :
$$\frac{r_n}{\mu_{2r}} \sum_{i=1}^{N_t^n - 1} \Delta(n,i)^{2-r} |\Delta_i^n X|^{2r} \alpha(n,i) \xrightarrow{u.c.p.} \int_0^t \sigma_s^{2r}\, \widetilde{a}(2)_s\, ds. \tag{28}$$

We start by deriving a central limit theorem for U_t^n. Notice that U_t^n can be expressed as:

$$\begin{aligned}
\mathsf{U}_t^n &= \sqrt{r_n} \sum_{i=1}^{N_t^n - 1} \Delta(n,i) \left(|\beta_i^n|^r - \left|\beta_{i+1}'^n\right|^r \right) \alpha(n,i) = \sqrt{r_n} \left[\sum_{i=1}^{N_t^n - 1} \Delta(n,i) |\beta_i^n|^r\, \alpha(n,i) - \sum_{i=1}^{N_t^n - 1} \Delta(n,i) \left|\beta_{i+1}'^n\right|^r \alpha(n,i) \right] \\
&= \sqrt{r_n} \left[\sum_{i=1}^{N_t^n - 1} \Delta(n,i) |\beta_i^n|^r\, \alpha(n,i) - \sum_{i=2}^{N_t^n} \Delta(n,i-1) \left|\beta_i'^n\right|^r\, \alpha(n,i-1) \right] \\
&= \sqrt{r_n} \left[\sum_{i=1}^{N_t^n - 1} \Delta(n,i) |\beta_i^n|^r\, \alpha(n,i) - \sum_{i=1}^{N_t^n - 1} \Delta(n,i-1) \left|\beta_i'^n\right|^r\, \alpha(n,i-1) \right] + \mathsf{b}_{n,t}^{(1)}.
\end{aligned} \tag{29}$$

where, using the (immaterial) border condition $\alpha(n,0) = 0$, we have $\mathsf{b}_{n,t}^{(1)} = -\sqrt{r_n}\, \Delta\left(n, N_t^n - 1\right) \left|\beta'_{N_t^n}\right| \alpha\left(n, N_t^n - 1\right)$ and, due to Assumption $\mathcal{B}_2$, $\mathsf{b}_{n,t}^{(1)} \underset{n\to\infty}{\xrightarrow{\text{ucp}}} 0$ is a negligible border term. Whence

$$\mathsf{U}_t^n = \sqrt{r_n} \sum_{i=1}^{N_t^n - 1} \left[\Delta(n,i) \left(|\beta_i^n|^r - \mu_r\, \sigma_{t(n,i-1)}^r \right) \alpha(n,i) - \Delta(n,i-1) \left(\left|\beta_i'^n\right|^r - \mu_r\, \sigma_{t(n,i-2)}^r \right) \alpha(n,i-1) \right] +$$

$$+\underbrace{\sqrt{r_n}\,\mu_r\sum_{i=1}^{N_t^n-1}\Big[\Delta(n,i)\,\sigma^r_{t(n,i-1)}\,\alpha(n,i)-\Delta(n,i-1)\,\sigma^r_{t(n,i-2)}\alpha(n,i-1)\Big]}_{\mathsf{b}^{(2)}_{n,t}}+\mathsf{b}^{(1)}_{n,t} \tag{30}$$

and the term $\mathsf{b}^{(2)}_{n,t}$ can be written as

$$\begin{aligned}\mathsf{b}^{(2)}_{n,t} &= \sqrt{r_n}\,\mu_r\sum_{i=1}^{N_t^n-1}\Delta(n,i)\,\sigma^r_{t(n,i-1)}\,\alpha(n,i)-\sqrt{r_n}\,\mu_r\sum_{i=0}^{N_t^n-2}\Delta(n,i)\,\sigma^r_{t(n,i-1)}\alpha(n,i)\\ &= \sqrt{r_n}\,\mu_r\,\Delta\left(n,N_t^n-1\right)\,\sigma^r_{t(n,N_t^n-2)}\,\alpha\left(n,N_t^n-1\right)\xrightarrow[n\to\infty]{\text{ucp}}0.\end{aligned}$$

where we have used, again, the border condition $\alpha(n,0)=0$. In summary

$$\mathsf{U}^n_t=\sum_{i=1}^{N_t^n}\left(\xi^n_i-\xi'^n_i\right)+\mathsf{b}_{t,n},$$

where

$$\begin{aligned}\xi^n_i &= \sqrt{r_n}\Delta(n,i)\left(|\beta^n_i|^r-\mu_r\left|\sigma_{t(n,i-1)}\right|^r\right)\alpha(n,i),\\ \xi'^n_i &= \sqrt{r_n}\Delta(n,i-1)\left(|\beta'^n_i|^r-\mu_r\left|\sigma_{t(n,i-2)}\right|^r\right)\alpha(n,i-1),\end{aligned}$$

and $\mathsf{b}_{t,n}=\mathsf{b}^{(1)}_{t,n}+\mathsf{b}^{(2)}_{t,n}$ is an asymptotically negligible reminder containing the border terms. Hence, in order to obtain a central limit theorem for U^n_t, it is enough to derive a central limit theorem for $\sum_{i=1}^{N_t^n}\zeta^n_i$, where $\zeta^n_i=(\xi^n_i-\xi'^n_i)$.

Consider the σ-fields $\mathcal{F}'_i=\mathcal{F}_{t(n,i)}\vee\sigma\left(\Delta(n,i+1)\right)$ and denote by $\mathbb{E}'_{i-1}[\cdot]$ the conditional expectation with respect to $\mathcal{F}'_{i-1}$. As pointed out by Hayashi et al. (2011), in order to prove $\mathcal{F}^0$-stable convergence we can apply Theorem IX.7.13 of Jacod and Shiryaev (2003). Since $\mathcal{F}'_{i-1}=\mathcal{F}_{t(n,i-1)}\vee\sigma\left(t(n,i)-t\left(n,i-1\right)\right)$ and since, under the null, W and $\Delta(n,i)$ are independent, it holds that $\mathbb{E}'_{i-1}\left[|\Delta^n_iW|^r\right]=\Delta\left(n,i\right)^{-r/2}\mu_r$, whence $\mathbb{E}'_{i-1}\left[\xi^n_i\right]=\mathbb{E}'_{i-1}\left[\xi'^n_i\right]=0$ and so $\mathbb{E}'_{i-1}\left[\zeta^n_i\right]=0$. Consequently it is enough to prove the following properties:

$$\begin{aligned}&\sum_{i=1}^{N_t^n}\mathbb{E}'_{i-1}\left[(\zeta^n_i)^2\right]\xrightarrow{P}2\left(\mu_{2r}-\mu_r^2\right)\int_0^t\sigma^{2r}_s\,\widetilde{a}(2)_s\,ds,\\ &\sum_{i=1}^{N_t^n}\mathbb{E}'_{i-1}\left[(\zeta^n_i)^2\,\mathbf{1}_{\{|\zeta^n_i|>\varepsilon\}}\right]\xrightarrow{P}0,\\ &\sum_{i=1}^{N_t^n}\mathbb{E}'_{i-1}\left[\zeta^n_i\Delta_iM\right]\xrightarrow{P}0,\end{aligned} \tag{31}$$

for all t and $\varepsilon>0$ and M is either a bounded $\mathcal{F}^0$-martingal orthogonal to W or $M=W$.

We have:

$$\mathbb{E}'_{i-1}\left[(\zeta^n_i)^2\right]=\mathbb{E}'_{i-1}\left[(\xi^n_i)^2+(\xi'^n_i)^2-2\xi^n_i\xi'^n_i\right]=\mathbb{E}'_{i-1}\left[(\xi^n_i)^2+(\xi'^n_i)^2\right],$$

where the second equality follows from the fact that $\xi^n_i\xi'^n_i=0$ by construction, since

$$\alpha(n,i)\alpha(n,i-1)=\mathbf{1}_{\{\Delta(n,i)>\Delta(n,i+1)\}}\mathbf{1}_{\{\Delta(n,i-1)\le\Delta(n,i)\}}\mathbf{1}_{\{\Delta(n,i-1)>\Delta(n,i)\}}\mathbf{1}_{\{\Delta(n,i-2)\le\Delta(n,i-1)\}}=0.$$

By construction, $\alpha(n,i)^k = \alpha(n,i)$ for all integers $k \in \mathbb{N}$. Consequently, we have:

$$\mathbb{E}'_{i-1}\left[(\xi_i^n)^2\right] = r_n \Delta(n,i)^2 \left(\mu_{2r} - \mu_r^2\right) \left|\sigma_{t(n,i-1)}\right|^{2r} \alpha(n,i).$$

Since σ is cadlag, Assumption $\mathcal{B}_2$ (with $q = 2$) implies:

$$\sum_{i=1}^{N_t^n} \mathbb{E}'_{i-1}\left[(\xi_i^n)^2\right] \underset{n\to\infty}{\overset{\text{ucp}}{\longrightarrow}} \left(\mu_{2r} - \mu_r^2\right) \int_0^t \sigma_s^{2r}\, \widetilde{a}(2)_s \, ds.$$

Analogously, we obtain that $\sum_{i=1}^{N_t^n} \mathbb{E}'_{i-1}\left[(\xi_i'^n)^2\right]$ converges in probability to the same limit, which implies:

$$\sum_{i=1}^{N_t^n} \mathbb{E}'_{i-1}\left[(\zeta_i^n)^2\right] \overset{P}{\longrightarrow} 2\left(\mu_{2r} - \mu_r^2\right) \int_0^t \sigma_s^{2r}\, \widetilde{a}(2)_s \, ds.$$

We can prove the convergence $\sum_{i=1}^{N_t^n} \mathbb{E}'_{i-1}\left[(\zeta_i^n)^2 \mathbf{1}_{\{|\zeta_i^n|>\varepsilon\}}\right] \overset{P}{\longrightarrow} 0$ by noticing that first that $(\xi_i^n)^2 (\xi_i'^n)^2 = 0$, whence $(\zeta_i^n)^4 = (\xi_i^n)^4 + (\xi_i'^n)^4$. Now consider that

$$\mathbb{E}'_{i-1}\left[(\xi_i^n)^4\right] = r_n^2\, \Delta(n,i)^4\, \sigma_{t(n,i-1)}^{4\,r} \left(\mu_{4\,r} + \mu_r^4 - 4\,\mu_{3\,r}\,\mu_r + 6\,\mu_{2\,r}\,\mu_r^2 - 4\,\mu_r^4\right)$$

whence Assumption $\mathcal{B}_2$ implies $\sum_{i=1}^{N_t^n} \mathbb{E}'_{i-1}\left[(\xi_i^n)^4\right] \underset{n\to\infty}{\overset{\text{ucp}}{\longrightarrow}} 0$ and, similarly, $\sum_{i=1}^{N_t^n} \mathbb{E}'_{i-1}\left[(\xi_i'^n)^4\right] \underset{n\to\infty}{\overset{\text{ucp}}{\longrightarrow}} 0$, from which

$$\sum_{i=1}^{N_t^n} \mathbb{E}'_{i-1}\left[(\zeta_i^n)^4\right] \underset{n\to\infty}{\overset{\text{ucp}}{\longrightarrow}} 0$$

and the second of the conditions in (31) is satisfied.

Besides, mirroring the same argument used in the proof of Theorem Hayashi et al. (2011), it can be easily seen that $\mathbb{E}'_{i-1}\left[\xi_i^n \Delta_i W\right] = \mathbb{E}'_{i-1}\left[\xi_i'^n \Delta_i W\right] = 0$. In fact $\mathbb{E}'_{i-1}\left[\xi_i^n \Delta_i W\right] = \mathbb{E}'_{i-1}\left[\xi_i'^n \Delta_i W\right] \sim \mathbb{E}\left[|u|\; u\right] = 0$ with u distributed as a $\mathcal{N}(0,1)$.

Finally, for any martingale M such that $[M,W] \equiv 0$, the identity $\mathbb{E}'_{i-1}\left[\xi_i^n \Delta_i M\right] = \mathbb{E}'_{i-1}\left[\xi_i'^n \Delta_i M\right] = 0$ follows from the same argument of Example 2 in Podolskij and Vetter (2010) (and, again, the independence between W and $\Delta(n,i)$), which completes the proof of (26).

In order to prove the asymptotic negligibility of the approximation error we notice that

$$\left|\sqrt{r_n} \sum_{i=1}^{N_t^n-1} \Delta(n,i) \left(|\beta_i^n|^r - \Delta(n,i)^{-\frac{r}{2}} \left|\Delta_i^n X\right|^r\right) \alpha(n,i)\right| \leq \sqrt{r_n} \sum_{i=1}^{N_t^n-1} \Delta(n,i) \left||\beta_i^n|^r - \Delta(n,i)^{-\frac{r}{2}} \left|\Delta_i^n X\right|^r\right|.$$

The right-hand side of the above inequality is asymptotically negligible as shown in the proof of Theorem 3.2 of Hayashi et al. (2011). Analogously, it is possible to show that the same convergence holds when β_i^n is replaced by $\beta_i'^n$, which completes the proof of the convergence (27).

Now, to complete the proof we have to prove the equation (28). Consider an approximation:

$$r_n \sum_{i=1}^{N_t^n-1} \Delta(n,i)^2 \left|\beta_i^n\right|^{2r} \alpha(n,i) = \sum_{i=1}^{N_t^n-1} \eta_i^n,$$

where $\eta_i^n = r_n \Delta(n,i)^2 \left|\beta_i^n\right|^{2r} \alpha(n,i)$. We have $\mathbb{E}'_{i-1}\left[\eta_i^n\right] = r_n \Delta(n,i)^2 \mu_{2r} \left|\sigma_{t(n,i-1)}\right|^{2r} \alpha(n,i)$, and

$$\mathbb{E}'_{i-1}\left[\left(\eta_i^n - \mathbb{E}'_{i-1}\left[\eta_i^n\right]\right)^2\right] = \left(\mu_{4r} - \mu_{2r}^2\right) r_n^2 \Delta(n,i)^4 \left|\sigma_{t(n,i-1)}\right|^{4r} \alpha(n,i).$$

Consequently, by Assumption $\mathcal{B}_2$,

$$\sum_{i=1}^{N_t^n - 1} \mathbb{E}'_{i-1}\left[\left(\eta_i^n - \mathbb{E}'_{i-1}\left[\eta_i^n\right]\right)^2\right] \stackrel{P}{\longrightarrow} 0,$$

which implies that

$$\sum_{i=1}^{N_t^n - 1} \left(\eta_i^n - \mathbb{E}'_{i-1}\left[\eta_i^n\right]\right) \stackrel{u.c.p}{\longrightarrow} 0.$$

On the other hand, using Assumption $\mathcal{B}_2$ again, we obtain:

$$\sum_{i=1}^{N_t^n - 1} \mathbb{E}'_{i-1}\left[\eta_i^n\right] = \sum_{i=1}^{N_t^n - 1} r_n \Delta(n,i)^2 \mu_{2r} \left|\sigma_{t(n,i-1)}\right|^{2r} \alpha(n,i) \stackrel{u.c.p}{\longrightarrow} \mu_{2r} \int_0^t \sigma_s^{2r}\, \widetilde{a}(2)_s \, ds.$$

Hence, it remains to prove that

$$r_n \sum_{i=1}^{N_t^n - 1} \left(\Delta(n,i)^2 \left|\beta_i^n\right|^{2r} - \Delta(n,i)^{2-r} \left|\Delta_i^n X\right|^{2r}\right) \alpha(n,i) \stackrel{u.c.p}{\longrightarrow} 0,$$

which follows from the proof of Theorem 3.1 of Hayashi et al. (2011) (in particular, form equations (6.8) and (6.9) in the proof with $j = 1$, $q = 1$ and $p = 2$). □

Proof of Theorem 3.2 .

The proof follows from the same arguments as in Barndorff-Nielsen et al. (2006) and Mancini (2009) and omitted for brevity. □

A.2 Proofs of results in presence of microstructural noise

Proof of Theorem 4.1.

The proof consists of the three steps:

1. the proof of the convergence in distribution of the numerator of the test statistic :

$$\mathsf{U}_t^n = \sqrt{n} \sum_{i=1}^{N_t - k_n} \Delta_i^n Y \left(\Delta(n,i) - \frac{1}{k_n} \sum_{j=0}^{k_n - 1} \Delta(n, i+j)\right) \stackrel{weakly}{\longrightarrow} \sqrt{m_2 - m_1^2} \int_0^t u_s v_s \, dW'_s = \mathsf{U}_t.$$

2. The proof of the convergence in probability of the estimator of the variance:

$$\mathsf{V}_t^n = n \sum_{i=1}^{N_t - k_n} \left(\Delta_i^n Y \left(\Delta(n,i) - \frac{1}{k_n} \sum_{j=0}^{k_n - 1} \Delta(n, i+j)\right)\right)^2 \stackrel{p}{\longrightarrow} \left(m_2 - m_1^2\right) \int_0^t \left(u_s v_s\right)^2 \, ds = \mathsf{V}_t.$$

3. The proof of the joint convergence in distribution:

$$\left(\mathsf{U}_t^n, \mathsf{V}_t^n\right)' \stackrel{weakly}{\longrightarrow} \left(\mathsf{U}_t, \mathsf{V}_t\right)',$$

which allows to conclude that $\mathsf{U}_t^n/\sqrt{\mathsf{V}_t^n} \overset{weakly}{\longrightarrow} \mathsf{U}_t/\sqrt{\mathsf{V}_t}$ and completes the proof.

To simplify the notations, we set $m = N_t^n - k_n$ everywhere below. Notice that under the assumptions of Theorem 4.1 it holds that $m = O_p(n)$.

Step 1. Consider the decomposition

$$\sum_{i=1}^{m} \Delta_i^n Y \left(\Delta(n,i) - \frac{1}{k_n} \sum_{j=0}^{k_n-1} \Delta(n,i+j) \right) = \mathsf{A}_1 + \mathsf{A}_2 + \mathsf{A}_3 + \mathsf{A}_4,$$

where

$$\begin{aligned}
\mathsf{A}_1 &= \sum_{i=1}^{m} \Delta_i^n U \,\widetilde{\Delta}_i, \\
\mathsf{A}_2 &= \sum_{i=1}^{m} \Delta_i^n X \,\widetilde{\Delta}_i, \\
\mathsf{A}_3 &= \sum_{i=1}^{m} \Delta_i^n U \left(\mathbb{E}\left[\Delta(n,i) \mid \mathcal{F}_{i-1}\right] - \frac{1}{k_n} \sum_{j=0}^{k_n-1} \Delta(n,i+j) \right), \\
\mathsf{A}_4 &= \sum_{i=1}^{m} \Delta_i^n X \left(\mathbb{E}\left[\Delta(n,i) \mid \mathcal{F}_{i-1}\right] - \frac{1}{k_n} \sum_{j=0}^{k_n-1} \Delta(n,i+j) \right).
\end{aligned}$$

and $\widetilde{\Delta}_i = \Delta(n,i) - \mathbb{E}\left[\Delta(n,i) \mid \mathcal{F}_{i-1}\right]$ and we have used $Y = X + U$. The leading term is A_1. We prove that $\sqrt{n}\mathsf{A}_1$ converges in law and, for $i = 2,3,4$, $\sqrt{n}\mathsf{A}_i$ are asymptotically negligible.

First, consider $\sqrt{n}\,\mathsf{A}_1$. Write:

$$\sqrt{n}\,\mathsf{A}_1 = \sum_{i=1}^{m} \zeta_i^n,$$

where $\zeta_i^n = \sqrt{n}\,\Delta_i^n U \,\widetilde{\Delta}_i$. In order to condition on the noise sequence, we defined the σ-fields $\mathcal{F}_i^U = \mathcal{F}_{t(n,i)} \vee \sigma\left(\Delta_{i+1}^n U\right)$. Then, we have:

$$\sum_{i=1}^{m} \mathbb{E}\left[\zeta_i^n \mid \mathcal{F}_{i-1}^U\right] = 0,$$

since $\widetilde{\Delta}_i$ is a martingale difference independent from $\Delta_i^n U$, which is adapted to $\mathcal{F}_{i-1}^U$.

Next,

$$\sum_{i=1}^{m} \mathbb{E}\left[(\zeta_i^n)^2 \,\middle|\, \mathcal{F}_{i-1}^U \right] = \sum_{i=1}^{m} (m_2 - m_1^2) \left(\Delta_i^n U\right)^2 \left(v_{t(n,i-1)}^n\right)^2 \frac{1}{n} \overset{p}{\longrightarrow} (m_2 - m_1^2) \int_0^t u_s^2 v_s^2 \, ds.$$

Finally, we have:

$$\sum_{i=1}^{m} \mathbb{E}\left[(\zeta_i^n)^4 \,\middle|\, \mathcal{F}_{i-1}^U \right] = \sum_{i=1}^{m} \left(\Delta_i^n U\right)^4 \left(v_{t(n,i-1)}^n\right)^4 \mathbb{E}\left[\left(\varepsilon(n,i) - m_1\right)^4 \right] \frac{1}{n^2} \overset{p}{\longrightarrow} 0.$$

Consequently,

$$\sum_{i=1}^{m} \zeta_i^n \overset{weakly}{\longrightarrow} \sqrt{m_2 - m_1^2} \int_0^t u_s v_s \, dW_s'.$$

Now, consider $\sqrt{n}\,\mathsf{A}_2$. We have:

$$\sqrt{n}\,\mathsf{A}_2 = \sum_{i=1}^{m} \zeta_i^n(2),$$

where $\zeta_i^n(2) = \sqrt{n}\,\Delta_i^n X\,\widetilde{\Delta}_i$. By conditioning on $\Delta(n,i)$ and using the estimate in (11), we obtain:

$$
\begin{aligned}
\left|\sum_{i=1}^m \mathbb{E}\left[\sqrt{n}\Delta_i^n X\tilde{\Delta}_i \;\middle|\; \mathcal{F}_{i-1}\right]\right| &= \left|\sum_{i=1}^m \mathbb{E}_{i-1}\left[\mathbb{E}\left[\sqrt{n}\Delta_i^n X\tilde{\Delta}_i \;\middle|\; \mathcal{F}_{i-1}\vee\sigma(\Delta(n,i))\right]\right]\right| \\
&\le \sum_{i=1}^m \mathbb{E}_{i-1}\left[\left|\mathbb{E}\left[\sqrt{n}\,\Delta_i^n X\,\tilde{\Delta}_i \;\middle|\; \mathcal{F}_{i-1}\vee\sigma(\Delta(n,i))\right]\right|\right] \\
&= \sum_{i=1}^m \mathbb{E}_{i-1}\left[\left|\sqrt{n}\,\tilde{\Delta}_i\mathbb{E}\left[\Delta_i^n X \mid \mathcal{F}_{i-1}\vee\sigma(\Delta(n,i))\right]\right|\right] \\
&\le \sum_{i=1}^m \mathbb{E}_{i-1}\left[\sqrt{n}\left|\tilde{\Delta}_i\right|\Delta(n,i)^\alpha\right] \\
&= \sum_{i=1}^m \mathbb{E}_{i-1}\left[\sqrt{n}\left|\Delta(n,i)-\frac{1}{n}v^n_{t(n,i-1)}m_1\right|\frac{1}{n^\alpha}(v^n_{t(n,i-1)})^\alpha\varepsilon(n,i)^\alpha\right] \\
&= \sum_{i=1}^m \mathbb{E}_{i-1}\left[\sqrt{n}\left|\frac{1}{n}v_{t(n,i-1)}\varepsilon(n,i)-\frac{1}{n}v^n_{t(n,i-1)}m_1\right|\frac{1}{n^\alpha}(v^n_{t(n,i-1)})^\alpha\varepsilon(n,i)^\alpha\right] \\
&= \sum_{i=1}^m \mathbb{E}_{i-1}\left[\sqrt{n}\left|\varepsilon(n,i)-m_1\right|\frac{1}{n^{\alpha+1}}(v^n_{t(n,i-1)})^{1+\alpha}\,\varepsilon(n,i)^\alpha\right] \\
&= \sum_{i=1}^m \sqrt{n}\mathbb{E}\left[\left|\varepsilon(n,i)^{1+\alpha}-\varepsilon(n,i)^\alpha m_1\right|\right]\frac{1}{n^{\alpha+1}}(v^n_{t(n,i-1)})^{1+\alpha} \\
&\le C\,\sqrt{n}\frac{1}{n^{\alpha+1}}\sum_{i=1}^m\left(\mathbb{E}\left[\varepsilon(n,i)^{1+\alpha}\right]+\mathbb{E}\left[\varepsilon(n,i)^\alpha\right]m_1\right).
\end{aligned}
$$

Consequently,

$$
\left|\sum_{i=1}^m \mathbb{E}\left[\zeta_i^n(2)\mid\mathcal{F}_{i-1}\right]\right| = O_p\left(n^{\frac{1}{2}-\alpha}\right)\xrightarrow{p} 0,
$$

having used $\alpha > \frac{1}{2}$ as in Assumption $\mathcal{E}_2$ and $\sum_{i=1}^m\left(\mathbb{E}\left[\varepsilon(n,i)^{1+\alpha}\right]+\mathbb{E}\left[\varepsilon(n,i)^\alpha\right]m_1\right) = O_p(n)$. Similarly, using the estimate in (12), we obtain:

$$
\mathbb{E}\left[\zeta_i^n(2)^2 \;\middle|\; \mathcal{F}_{i-1}\right] \le C\,n\,\mathbb{E}\left[\widetilde{\Delta}_i^2\Delta(n,i)^{2\alpha/2} \;\middle|\; \mathcal{F}_{i-1}\right] = C\,n^{-1-\alpha}.
$$

Consequently,

$$
\sum_{i=1}^m \mathbb{E}\left[(\zeta_i^n(2))^2 \;\middle|\; \mathcal{F}_{i-1}\right] = O_p\left(n^{-\alpha}\right)\xrightarrow{p} 0,
$$

which implies, using Lemma 4.1 in Jacod (2012), that $\sqrt{n}\,\mathsf{A}_2$ is asymptotically negligible. Next, consider $\sqrt{n}\,\mathsf{A}_3$, which can be re-written as

$$
\sqrt{n}\,\mathsf{A}_3 = \sum_{i=1}^m \zeta_i^n(3),
$$

where $\zeta_i^n(3) = \sqrt{n}\,\Delta_i^n U\left(\mathbb{E}\left[\Delta(n,i)\mid\mathcal{F}_{i-1}\right]-\frac{1}{k_n}\sum_{j=0}^{k_n-1}\Delta(n,i+j)\right)$. Using the definition of $\Delta(n,i)$'s we obtain the decomposition:

$$
\zeta_i^n(3) = \sqrt{n}\,\Delta_i^n U\left(\frac{1}{k_n}\sum_{j=0}^{k_n-1}\mathbb{E}\left[\Delta(n,i)\mid\mathcal{F}_{i-1}\right]-\frac{1}{k_n}\sum_{j=0}^{k_n-1}\Delta(n,i+j)\right)
$$

$$= \sqrt{n}\,\Delta_i^n U \left(\frac{1}{k_n} \sum_{j=0}^{k_n-1} \left(\frac{1}{n} v^n_{t(n,i-1)} m_1 - \frac{1}{n} v^n_{t(n,i+j-1)} \varepsilon(n,i+j) \right) \right) = \zeta_i^n(3,1) + \zeta_i^n(3,2), \tag{32}$$

where

$$\begin{aligned} \zeta_i^n(3,1) &= \sqrt{n}\,\Delta_i^n U \left(\frac{1}{k_n} \sum_{j=0}^{k_n-1} \left(\frac{1}{n} v^n_{t(n,i-1)} \left(m_1 - \varepsilon(n,i+j) \right) \right) \right), \\ \zeta_i^n(3,2) &= \sqrt{n}\,\Delta_i^n U \left(\frac{1}{k_n} \sum_{j=0}^{k_n-1} \left(\frac{1}{n} \left(v^n_{t(n,i-1)} - v^n_{t(n,i+j-1)} \right) \varepsilon(n,i+j) \right) \right). \end{aligned}$$

So, it is enough to show the two sums, $\sum_{i=1}^{m} \zeta_i^n(3,1)$ and $\sum_{i=1}^{m} \zeta_i^n(3,2)$, converge to zero in probability. Concerning the first sum we notice that, since $\Delta_i^n U$ is independent from the sampling times and $\mathbb{E}\left[\Delta_i^n U\right] = 0$, we have

$$\mathbb{E}\left[\sum_{i=1}^{m} \zeta_i^n(3,1) \right] = \mathbb{E}\left[\sum_{i=1}^{m} \zeta_i^n(3,2) \right] = 0.$$

Next, we notice that

$$\mathbb{E}\left[\left(\sum_{i=1}^{m} \zeta_i^n(3,1) \right)^2 \right] = \mathbb{E}\left[\sum_{i=1}^{m} \zeta_i^n(3,1)^2 + 2 \sum_{i=1}^{m-1} \zeta_i^n(3,1) \zeta_{i+1}^n(3,1) + \ldots + 2 \sum_{i=1}^{m-(k_n-1)} \zeta_i^n(3,1) \zeta_{i+k_n-1}^n(3,1) \right].$$

Using the independence of $\Delta_i^n U$'s from the sampling times and the boundedness of $v^n_{t(n,i-1)}$, for some constant $C > 0$, we obtain

$$\begin{aligned} \mathbb{E}\left[\zeta_i^n(3,1)^2\right] &= \mathbb{E}\left[\frac{n\,(\Delta_i^n U)^2}{k_n^2\, n^2} \left(\sum_{j=0}^{k_n-1} \left(v^n_{t(n,i-1)} \left(m_1 - \varepsilon(n,i+j) \right) \right) \right)^2 \right] \le C\, \frac{\mathbb{E}\left[(\Delta_i^n U)^2 \right]}{k_n^2\, n} \mathbb{E}\left[\sum_{j=0}^{k_n-1} \left(m_1 - \varepsilon(n,i+j) \right)^2 \right] \\ &= C\, \frac{1}{k_n^2 n} k_n (m_2 - m_1^2), \end{aligned}$$

and[2]

$$\begin{aligned} &\left| \mathbb{E}\left[\zeta_i^n(3,1) \zeta_{i+1}^n(3,1) \right] \right| = \\ &n\,\mathbb{E}\left[\left| \Delta_i^n U \Delta_{i+1}^n U \right| \right] \mathbb{E}\left[\left| \left(\frac{1}{k_n} \sum_{j=0}^{k_n-1} \left(\frac{1}{n} v^n_{t(n,i-1)} \left(m_1 - \varepsilon(n,i+j) \right) \right) \right) \left(\frac{1}{k_n} \sum_{j=0}^{k_n-1} \left(\frac{1}{n} v^n_{t(n,i)} \left(m_1 - \varepsilon(n,i+1+j) \right) \right) \right) \right| \right] \\ \le\ & C\, n\, \mathbb{E}\left[\left| \Delta_i^n U \Delta_{i+1}^n U \right| \right] \frac{1}{k_n^2\, n^2} \mathbb{E}\left[\left(\sum_{j=0}^{k_n-1} \left(m_1 - \varepsilon(n,i+j) \right) \right)^2 \right] = C\, \frac{1}{k_n^2 n} k_n (m_2 - m_1^2). \end{aligned}$$

For $k \ge 2$, $\mathbb{E}\left[\zeta_i^n(3,1) \zeta_{i+k}^n(3,1) \right] = 0$, since $\mathbb{E}\left[\Delta_i^n U \Delta_{i+k}^n U \right] = 0$. Consequently, $\mathbb{E}\left[\left(\sum_{i=1}^{m} \zeta_i^n(3,1) \right)^2 \right] \sim k_n^{-1}$ converges to

[2] We use

$$\mathsf{cov}\left(X, Y \right) \le \max\left(\mathsf{var}\left(\sigma_X^2 \right), \mathsf{var}\left(\sigma_Y^2 \right) \right), \tag{33}$$

and the boundedness of v^n. We do the same in deriving the inequality in (34).

zero, which implies that

$$\sum_{i=1}^{m} \zeta_i^n(3,1) \xrightarrow{p} 0.$$

Now, we consider the second sum $\sum_{i=1}^{m} \zeta_i^n(3,2)$ and notice that

$$\mathbb{E}\left[\left(\sum_{i=1}^{m} \zeta_i^n(3,2)\right)^2\right] = \sum_{i=1}^{m} \mathbb{E}\left[\zeta_i^n(3,2)^2\right] + 2\sum_{i=1}^{m-1} \mathbb{E}\left[\zeta_i^n(3,2)\zeta_{i+1}^n(3,2)\right] \leq C \sum_{i=1}^{m} \mathbb{E}\left[\zeta_i^n(3,2)^2\right] \tag{34}$$

We have

$$\begin{aligned}
\mathbb{E}\left[\zeta_i^n(3,2)^2\right] &= n\,\mathbb{E}\left[(\Delta_i^n U)^2\right]\mathbb{E}\left[\left(\frac{1}{n\,k_n}\sum_{j=0}^{k_n-1}\left(\left(v_{t(n,i-1)}^n - v_{t(n,i+j-1)}^n\right)\varepsilon(n,i+j)\right)\right)^2\right] \\
&= \frac{\mathbb{E}\left[(\Delta_i^n U)^2\right]}{nk_n^2}\left(\sum_{j=0}^{k_n-1}\mathbb{E}\left[\left(\left(v_{t(n,i-1)}^n - v_{t(n,i+j-1)}^n\right)\varepsilon(n,i+j)\right)^2\right]\right. \\
&+ \left.\underbrace{2\sum_{j=0}^{k_n-1}\sum_{k=1}^{j-1}\mathbb{E}\left[\left(v_{t(n,i-1)}^n - v_{t(n,i+j-1)}^n\right)\left(v_{t(n,i-1)}^n - v_{t(n,i+k-1)}^n\right)\varepsilon(n,i+j)\varepsilon(n,i+k)\right]}_{\mathsf{B}_n}\right).
\end{aligned}$$

Since v_t^n is a bounded semimartingale, we have the inequality:

$$\mathbb{E}\left[\left|v_{t(n,i-1)}^n - v_{t(n,i+j-1)}^n\right|^2 \,\middle|\, \mathcal{F}_{i-1}\right] \leq \mathbb{E}\left[t(n,i+j-1) - t(n,i-1) \mid \mathcal{F}_{i-1}\right] \leq C\,m_1\,\frac{j}{n},$$

so that, by the Jensen inequality,

$$\mathbb{E}\left[\left|v_{t(n,i-1)}^n - v_{t(n,i+j-1)}^n\right| \,\middle|\, \mathcal{F}_{i-1}\right] \leq C\sqrt{\frac{j}{n}}.$$

Consequently,

$$\sum_{j=0}^{k_n-1}\mathbb{E}\left[\left(\left(v_{t(n,i-1)}^n - v_{t(n,i+j-1)}^n\right)\varepsilon(n,i+j)\right)^2\right] \leq C\,m_2\,\frac{k_n(k_n-1)}{2\,n} \sim \frac{k_n^2}{n},$$

which, using the Cauchy-Schwartz inequality, gives

$$|\mathsf{B}_n| \leq C\sum_{j=0}^{k_n-1}\sum_{k=1}^{j-1}\sqrt{\frac{j\,k}{n^2}m_1^2} \leq C\sum_{j=0}^{k_n-1}\sum_{k=1}^{j-1}\sqrt{\frac{j^2}{n^2}m_1^2} \leq C\sum_{j=0}^{k_n-1}\frac{j^2}{n} = C\,\frac{k_n(k_n-1)(2k_n-1)}{6n} \sim \frac{k_n^3}{n},$$

which implies that $\mathbb{E}\left[\zeta_i^n(3,2)^2\right] = O\left(k_n/n^2\right)$. As a result, we have

$$\mathbb{E}\left[\left(\sum_{i=1}^{m}\zeta_i^n(3,2)\right)^2\right] \longrightarrow 0 \text{ and } \sum_{i=1}^{m}\zeta_i^n(3,2) \xrightarrow{p} 0.$$

Consequently, $\sqrt{n}\mathsf{A}_3$ converges to zero in probability. Finally, consider $\sqrt{n}\mathsf{A}_4$. Consider the decomposition

$$\sqrt{n}\mathsf{A}_4 = \sum_{i=1}^{m}\zeta_i^n(4,1) + \sum_{i=1}^{m}\zeta_i^n(4,2) + \sum_{i=1}^{m}\zeta_i^n(4,3),$$

where

$$\begin{aligned}
\zeta_i^n(4,1) &= \sqrt{n}\Delta_i X \frac{1}{k_n}\left(\mathbb{E}\left[\Delta(n,i) \mid \mathcal{F}_{i-1}\right] - \Delta(n,i)\right),\\
\zeta_i^n(4,2) &= \sqrt{n}\Delta_i X \frac{1}{k_n}\sum_{j=1}^{k_n-1}\left(\mathbb{E}\left[\Delta(n,i) \mid \mathcal{F}_{i-1}\right] - \mathbb{E}\left[\Delta(n,i+j) \mid \mathcal{F}_{i-1}\right]\right),\\
\zeta_i^n(4,3) &= \sqrt{n}\Delta_i X \frac{1}{k_n}\sum_{j=1}^{k_n-1}\left(\mathbb{E}\left[\Delta(n,i+j) \mid \mathcal{F}_{i-1}\right] - \Delta(n,i+j)\right).
\end{aligned}$$

Using Assumption $\mathcal{E}_2$ and boundedness of v^n, for the first term we obtain

$$\mathbb{E}\left[|\zeta_i^n(4,1)| \mid \mathcal{F}_{i-1}\right] \le C\frac{\sqrt{n}}{k_n}\mathbb{E}\left[\Delta(n,i)^{\alpha/2}\left|\mathbb{E}\left[\Delta(n,i) \mid \mathcal{F}_{i-1}\right] - \Delta(n,i)\right| \,\middle|\, \mathcal{F}_{i-1}\right] \le \frac{C}{n^{(1+\alpha)/2}k_n}.$$

A similar computation shows that

$$\mathbb{E}\left[|\zeta_i^n(4,3)| \mid \mathcal{F}_{i-1}\right] \le \frac{C}{n^{(1+\alpha)/2}}.$$

So we have

$$\sum_{i=1}^m \mathbb{E}\left[|\zeta_i^n(4,1)|\right] \xrightarrow{p} 0, \quad \sum_{i=1}^m \mathbb{E}\left[|\zeta_i^n(4,3)|\right] \xrightarrow{p} 0.$$

Finally, for the second term, notice that

$$\left|\mathbb{E}\left[\Delta(n,i) - \Delta(n,i+j) \mid \mathcal{F}_{i-1}\right]\right| = \frac{m_1}{n}\left|\mathbb{E}\left[v^n_{t(n,i-1)} - v^n_{t(n,i+j-1)} \,\middle|\, \mathcal{F}_{i-1}\right]\right| \le \frac{m_1}{n}\sum_{k=1}^{j}\Delta(n,i+k-1),$$

where we have used the (semimartingale) property $\left|\mathbb{E}\left[v^n_{t(n,i)} - v^n_{t(n,i-1)} \,\middle|\, \mathcal{F}_{i-1}\right]\right| \le C\,\Delta(n,i)$ and a telescopic sum. Consequently, using Assumption $\mathcal{E}_2$ and boundedness of v^n again, we obtain

$$\mathbb{E}\left[|\zeta_i^n(4,2)| \mid \mathcal{F}_{i-1}\right] \le \frac{C}{\sqrt{n}k_n}\mathbb{E}\left[\Delta(n,i)^{\alpha/2}\sum_{j=1}^{k_n-1}\sum_{k=1}^{j}\Delta(n,i+k-1) \,\middle|\, \mathcal{F}_{i-1}\right] \le C\,\frac{k_n(k_n+1)}{n^{(\alpha+1)/2+1}k_n},$$

which implies that

$$\sum_{i=1}^m \mathbb{E}\left[|\zeta_i^n(4,2)| \mid \mathcal{F}_{i-1}\right] \xrightarrow{p} 0.$$

This completes the proof of the first step.

Step 2. We now prove that

$$n\sum_{i=1}^m\left(\Delta_i^n Y\left(\Delta(n,i) - \frac{1}{k_n}\sum_{j=0}^{k_n-1}\Delta(n,i+j)\right)\right)^2 \xrightarrow{p} \left(m_2 - m_1^2\right)\int_0^t \left(u_s v_s\right)^2 ds.$$

Consider the decomposition

$$n\sum_{i=1}^m\left(\Delta_i^n Y\left(\Delta(n,i) - \frac{1}{k_n}\sum_{j=0}^{k_n-1}\Delta(n,i+j)\right)\right)^2 = \mathsf{B}_1 + \mathsf{B}_2 + 2\mathsf{B}_3,$$

where

$$
\begin{aligned}
\mathsf{B}_1 &= n\sum_{i=1}^{m}\left(\Delta_i^n Y\left(\Delta(n,i)-\frac{1}{n}v_{t(n,i-1)}^n m_1\right)\right)^2,\\
\mathsf{B}_2 &= n\sum_{i=1}^{m}\left(\Delta_i^n Y\left(\frac{1}{n}v_{t(n,i-1)}^n m_1-\frac{1}{k_n}\sum_{j=0}^{k_n-1}\Delta(n,i+j)\right)\right)^2,\\
\mathsf{B}_3 &= n\sum_{i=1}^{m}\left(\Delta_i^n Y\right)^2\left(\Delta(n,i)-\frac{1}{n}v_{t(n,i-1)}^n m_1\right)\left(\frac{1}{n}v_{t(n,i-1)}^n m_1-\frac{1}{k_n}\sum_{j=0}^{k_n-1}\Delta(n,i+j)\right).
\end{aligned}
$$

The decomposition $Y = X + U$ implies for B_1 the structure

$$
\mathsf{B}_1=\sum_{i=1}^{m}\left(b_i^n(1,1)+b_i^n(1,2)+2b_i^n(1,3)\right),
$$

where $b_i^n(1,1)$, $b_i^n(1,2)$ and $b_i^n(1,3)$ are given by

$$
\begin{aligned}
b_i^n(1,1) &= n\left(\Delta_i^n U\right)^2\left(\Delta(n,i)-\frac{1}{n}v_{t(n,i-1)}^n m_1\right)^2,\\
b_i^n(1,2) &= n\left(\Delta_i^n X\right)^2\left(\Delta(n,i)-\frac{1}{n}v_{t(n,i-1)}^n m_1\right)^2,\\
b_i^n(1,3) &= n\Delta_i^n U\Delta_i^n X\left(\Delta(n,i)-\frac{1}{n}v_{t(n,i-1)}^n m_1\right)^2.
\end{aligned}
$$

Note that $b_i^n(1,1)\geq 0$ and that

$$
\mathbb{E}\left[b_i^n(1,1)\mid\mathcal{F}_{i-1}^U\right]=\left(\Delta_i^n U\right)^2\left(m_2-m_1^2\right)\left(v_{t(n,i-1)}^n\right)^2\frac{1}{n}.
$$

By Assumption $\mathcal{E}_1$, we have the convergence

$$
\sum_{i=1}^{m}\left(\Delta_i^n U\right)^2\left(m_2-m_1^2\right)\left(v_{t(n,i-1)}^n\right)^2\frac{1}{n}\xrightarrow{p}\left(m_2-m_1^2\right)\int_0^t\left(u_s v_s\right)^2\,ds.
$$

On the other hand, it can be easily seen that

$$
\sum_{i=1}^{m}\mathbb{E}\left[\left(b_i^n(1,1)-\mathbb{E}\left[b_i^n(1,1)\mid\mathcal{F}_{i-1}\right]\right)^2\,\middle|\,\mathcal{F}_{i-1}\right]\xrightarrow{p}0.
$$

Consequently,

$$
\sum_{i=1}^{m}b_i^n(1,1)\xrightarrow{p}\left(m_2-m_1^2\right)\int_0^t\left(u_s v_s\right)^2\,ds.
$$

The other terms in B_1 are asymptotically negligible. Indeed, we have:

$$
\mathbb{E}\left[b_i^n(1,2)\mid\mathcal{F}_{i-1}\right]\leq C\,\frac{1}{n^{1+\alpha}}.
$$

Since, $b_i^n(1,2)$ is positive, this implies the negligibility of $\sum_{i=1}^{m}b_i^n(1,2)$. The asymptotic negligibility of $\sum_{i=1}^{m}b_i^n(1,3)$ follows from analogous arguments.

Next, analogously to the proof of the asymptotic negligibility of the term

$$\zeta_i^n(3) = \sqrt{n}\,\Delta_i^n U \left(\mathbb{E}\left[\Delta(n,i) \mid \mathcal{F}_{i-1}\right] - \frac{1}{k_n}\sum_{j=0}^{k_n-1} \Delta(n,i+j) \right)$$

defined in (32), we can conclude that B_2 is asymptotically negligible.

Finally, consider the decomposition:

$$\mathsf{B}_3 = \sum_{i=1}^{m} n\left(\Delta_i^n Y\right)^2 \frac{1}{n} v^n_{t(n,i-1)} \left(\varepsilon(n,i) - m_1\right) \left(\frac{1}{n} v^n_{t(n,i-1)} m_1 - \frac{1}{n\,k_n}\sum_{j=0}^{k_n-1} v^n_{t(n,i+j-1)}\varepsilon(n,i+j) \right) = \sum_{i=1}^{m}\left(b_i^n(3,1) + b_i^n(3,2)\right),$$

where

$$\begin{aligned} b_i^n(3,1) &= \frac{1}{n\,k_n}\left(\Delta_i^n Y\right)^2 \left(v^n_{t(n,i-1)}\right)^2 \left(\varepsilon(n,i) - m_1\right)\left(m_1 - \varepsilon(n,i)\right), \\ b_i^n(3,2) &= \frac{1}{n\,k_n}\left(\Delta_i^n Y\right)^2 v^n_{t(n,i-1)} \left(\varepsilon(n,i) - m_1\right) \sum_{j=1}^{k_n-1}\left(v^n_{t(n,i-1)} m_1 - v^n_{t(n,i+j-1)}\varepsilon(i+j) \right). \end{aligned}$$

For the first term we have

$$\sum_{i=i}^{m} \mathbb{E}\left[\left|b_i^n(3,1)\right| \mid \mathcal{F}_{i-1}\right] \le C \sum_{i=i}^{m} \mathbb{E}\left[\left(\varepsilon(n,i) - m_1\right)^2 \,\middle|\, \mathcal{F}_{i-1}\right] \frac{1}{nk_n} = C\sum_{i=i}^{m}\left(m_2 - m_1^2\right)\frac{1}{nk_n} \longrightarrow 0,$$

which implies that $\sum_{i=i}^{m} b_i^n(3,1)$ is asymptotically negligible. Next, since v^n is a semimartingale and $Y = X + U$ with X semimartingale and U bounded in probability, we have

$$\left| \mathbb{E}_{i-1}\left[v^n_{t(n,i-1)} - v^n_{t(n,i+j-1)} \right] \right| \le C\,\frac{j}{n}, \quad \left(\Delta_i^n Y\right)^2 = O_p(1)$$

whence

$$\begin{aligned} & \left| \sum_{i=1}^{m} \mathbb{E}\left[b_i^n(3,2) \mid \mathcal{F}_{i-1}\right] \right| = \\ = \;& \frac{1}{n\,k_n} \left| \sum_{i=1}^{m} \mathbb{E}_{i-1}\left[\left(\Delta_i^n Y\right)^2 v^n_{t(n,i-1)}\left(\varepsilon(n,i) - m_1\right) \sum_{j=1}^{k_n-1}\left(v^n_{t(n,i-1)} m_1 - v^n_{t(n,i+j-1)}\varepsilon(n,i+j)\right) \right] \right| \le \\ \le \;& C\,k_n^{-1}\,n^{-1} \left| \sum_{i=1}^{m}\left(\sum_{j=1}^{k_n-1} \mathbb{E}_{i-1}\left[m_1\left(v^n_{t(n,i-1)} - v^n_{t(n,i+j-1)} \right)\right] + \sum_{j=1}^{k_n-1}\mathbb{E}_{i-1}\left[v^n_{t(n,i+j-1)}\left(m_1 - \varepsilon(n,i+j)\right)\right] \right) \right| \\ \le \;& C\,k_n^{-1}\,n^{-1} \sum_{i=1}^{m}\sum_{j=1}^{k_n-1} \frac{j}{n} \le C\,\frac{k_n}{n} \xrightarrow{\;p\;} 0. \end{aligned}$$

where we have used $\mathbb{E}_{i-1}\left[v^n_{t(n,i+j-1)}\left(m_1 - \varepsilon(n,i+j)\right)\right] = 0$ which follows from the facts that $v^n_{t(n,i+j-1)}$ and $\varepsilon(n,i+j)$ are independent, with $\varepsilon(n,i)$ iid and $\mathbb{E}\left[\varepsilon(n,i)\right] = m_1$. Since $\left(\Delta_i^n Y\right)^4$ is bounded, using the boundedness of v^n and, again, the decomposition

$$v^n_{t(n,i-1)} m_1 - v^n_{t(n,i+j-1)}\varepsilon(n,i+j) = v^n_{t(n,i-1)} - v^n_{t(n,i+j-1)} + v^n_{t(n,i+j-1)}\left(m_1 - \varepsilon(n,i+j)\right) = \Delta\mathsf{v}^n_{i-1,j} + v^n_{t(n,i+j-1)}\left(m_1 - \varepsilon(n,i+j)\right)$$

where the last identity is definitional for $\Delta \mathsf{v}^n_{i-1,j}$. We obtain

$$\begin{aligned}
\mathbb{E}\left[b^n_i(3,2)^2 \mid \mathcal{F}_{i-1}\right] &\leq \frac{C}{n^2k_n^2}\mathbb{E}\left[\left(\sum_{j=1}^{k_n-1}\left(v^n_{t(n,i-1)}m_1 - v^n_{t(n,i+j-1)}\varepsilon(n,i+j)\right)\right)^2 \middle| \mathcal{F}_{i-1}\right] \\
&= \frac{C}{n^2k_n^2}\mathbb{E}\left[\left(\sum_{j=1}^{k_n-1}\left(\Delta \mathsf{v}^n_{i-1,j} + v^n_{t(n,i+j-1)}\left(m_1-\varepsilon(n,i+j)\right)\right)\right)^2 \middle| \mathcal{F}_{i-1}\right] \\
&= \frac{C}{n^2k_n^2}\mathbb{E}_{i-1}\left[\sum_{j=1}^{k_n-1}\left(\Delta \mathsf{v}^n_{i-1,j} + v^n_{t(n,i+j-1)}\left(m_1-\varepsilon(n,i+j)\right)\right)^2\right] + \\
&\quad + \frac{C}{n^2k_n^2}\mathbb{E}_{i-1}\left[\sum_{\ell>j}\left(\Delta \mathsf{v}^n_{i-1,j} + v^n_{t(n,i+j-1)}\left(m_1-\varepsilon(n,i+j)\right)\right)\left(\Delta \mathsf{v}^n_{i-1,\ell} + v^n_{t(n,i+\ell-1)}\left(m_1-\varepsilon(n,i+\ell)\right)\right)\right].
\end{aligned}$$

Since v^n and ε are independent, by conditioning upon the path of v^n we obtain

$$\begin{aligned}
\mathbb{E}\left[b^n_i(3,2)^2 \mid \mathcal{F}_{i-1}\right] &\leq \frac{C}{n^2k_n^2}\sum_{j=1}^{k_n-1}\mathbb{E}_{i-1}\left[\left(\Delta \mathsf{v}^n_{i-1,j}\right)^2 + \left(v^n_{t(n,i+j-1)}\right)^2\left(m_1-\varepsilon(n,i+j)\right)^2\right] + \frac{C}{n^2k_n^2}\sum_{\ell>j}\mathbb{E}_{i-1}\left[\Delta \mathsf{v}^n_{i-1,j}\,\Delta \mathsf{v}^n_{i-1,\ell}\right] \\
&\leq \frac{C}{n^2k_n^2}\left(\sum_{j=1}^{k_n-1}\frac{j}{n}+k_n\right) + \frac{C}{n^2k_n^2}\sum_{\ell>j}\mathbb{E}_{i-1}\left[\left(\Delta \mathsf{v}^n_{i-1,j}\right)^2 + \Delta \mathsf{v}^n_{i-1,j}\,\Delta \mathsf{v}^n_{i+j,\ell-j-1}\right] \\
&\leq \frac{C}{n^2k_n^2}\left(\frac{k_n\,(k_n-1)}{2\,n}+k_n\right) + \frac{C}{n^2k_n^2}\left(\sum_{j=1}^{k_n-1}\sum_{\ell=j+1}^{k_n-1}\frac{j}{n} + \sum_{j=1}^{k_n-1}\sum_{\ell=j+1}^{k_n-1}\mathbb{E}_{i-1}\left[\Delta \mathsf{v}^n_{i-1,j}\,\Delta \mathsf{v}^n_{i+j,\ell-j-1}\right]\right) \\
&\leq \frac{C}{n^2k_n^2}\left(\frac{k_n\,(k_n-1)}{2\,n}+k_n\right) + \frac{C}{n^2k_n^2}\left(\frac{k_n^3}{n}+\frac{k_n^4}{n^2}\right).
\end{aligned}$$

where we have used the semimartingale property of v after suitable conditioning inside the expectations and the summation rules

$$\sum_{j=1}^{k_n-1}\sum_{\ell=j+1}^{k_n-1} j = \frac{1}{6}\,k_n\,(k_n-1)\,(k_n-2)\,, \qquad \sum_{j=1}^{k_n-1}\sum_{\ell=j+1}^{k_n-1} j\,(\ell-j-1) = \frac{1}{24}\left(k_n^2-5\,k_n+6\right)\,k_n\,(k_n-1)$$

Consequently,

$$k_n\sum_{i=i}^{m}\mathbb{E}\left[\left(b^n_i(3,2)\right)^2 \middle| \mathcal{F}_{i-1}\right] \leq \frac{C}{n\,k_n}\left(\frac{k_n\,(k_n-1)}{2\,n}+k_n\right) + \frac{C}{n\,k_n}\left(\frac{k_n^3}{n}+\frac{k_n^4}{n^2}\right) \xrightarrow{\;p\;} 0,$$

which implies the asymptotic negligibility of $\sum_{i=i}^{m} b^n_i(3,2)$. Hence, B_3 is asymptotically negligible, which completes the proof of the second step.

Step 3. As well known, we have that the vector $(\mathsf{U}^n_t, \mathsf{V}^n_t)'$ converges in distribution to $(\mathsf{U}_t, \mathsf{V}_t)'$ if and only if for any real numbers a_1 and a_2,

$$a_1\mathsf{U}^n_t + a_2\mathsf{V}^n_t \overset{weakly}{\longrightarrow} a_1\mathsf{U}_t + a_2\mathsf{V}_t.$$

Previous steps shows that the leading terms in U^n_t and V^n_t are respectively $\sum_{i=1}^{m}\sqrt{n}\,\Delta^n_i U\left(\Delta(n,i) - \frac{1}{n}v^n_{t(n,i-1)}m_1\right)$ and $\sum_{i=1}^{m} n\left(\Delta^n_i U\right)^2\left(\Delta(n,i) - \frac{1}{n}v^n_{t(n,i-1)}m_1\right)^2$. Hence, it is enough to prove the convergence

$$\sum_{i=1}^{m}\xi^n_i \overset{weakly}{\longrightarrow} a_1\mathsf{U}_t + a_2\mathsf{V}_t,$$

where ξ_i^n is defined for generic a_1 and a_2 as

$$\xi_i^n = a_1 \sqrt{n}\, \Delta_i^n U \left(\Delta(n,i) - \frac{1}{n} v_{t(n,i-1)}^n m_1 \right) + a_2\, n\, (\Delta_i^n U)^2 \left(\Delta(n,i) - \frac{1}{n} v_{t(n,i-1)}^n m_1 \right)^2 .$$

Recall the definition $\mathcal{F}_i^U = \mathcal{F}_{t(n,i)} \vee \sigma\left(\Delta_{i+1}^n U\right)$, we have

$$\mathbb{E}\left[\xi_i^n \mid \mathcal{F}_{i-1}^U\right] = \frac{a_2}{n} (\Delta_i^n U)^2 \left(v_{t(n,i-1)}^n \right)^2 \left(m_2 - m_1^2\right),$$

which, by Assumption $\mathcal{E}_1$, implies:

$$\sum_{i=1}^m \mathbb{E}\left[\xi_i^n \mid \mathcal{F}_{i-1}^U\right] \xrightarrow{p} a_2 \left(m_2 - m_1^2\right) \int_0^t (u_s v_s)^2\, ds.$$

Next, consider the decomposition:

$$\sum_{i=1}^m \left(\mathbb{E}\left[(\xi_i^n)^2 \,\middle|\, \mathcal{F}_{i-1}^U \right] - \mathbb{E}\left[\xi_i^n \mid \mathcal{F}_{i-1}^U\right]^2 \right) = \sum_{i=1}^m \mathbb{E}\left[c_i^n(1) + c_i^n(2) + c_i^n(3) \mid \mathcal{F}_{i-1}^U\right],$$

where

$$\begin{aligned}
c_i^n(1) &= a_1^2\, n\, (\Delta_i^n U)^2 \left(\Delta(n,i) - \frac{1}{n} v_{t(n,i-1)}^n m_1 \right)^2, \\
c_i^n(2) &= a_2^2\, n^2\, (\Delta_i^n U)^4 \left(\Delta(n,i) - \frac{1}{n} v_{t(n,i-1)}^n m_1 \right)^4 - \left(\frac{a_2}{n}\right)^2 (\Delta_i^n U)^4 \left(v_{t(n,i-1)}^n \right)^4 \left(m_2 - m_1^2\right)^2, \\
c_i^n(3) &= 2 a_1 a_2\, n^{\frac{3}{2}}\, (\Delta_i^n U)^3 \left(\Delta(n,i) - \frac{1}{n} v_{t(n,i-1)}^n m_1 \right)^3 .
\end{aligned}$$

Assumption $\mathcal{E}_1$ implies that

$$\sum_{i=1}^m \mathbb{E}\left[c_i^n(1) \mid \mathcal{F}_{i-1}^U\right] \xrightarrow{p} a_1^2 \left(m_2 - m_1^2\right) \int_0^t (u_s v_s)^2\, ds,$$

while computations similar to those in *Step 2* show that $\sum_{i=1}^m \mathbb{E}\left[c_i^n(2) \mid \mathcal{F}_{i-1}^U\right] \xrightarrow{p} 0$. For the third term we have

$$\sum_{i=1}^m \left| \mathbb{E}\left[c_i^n(3) \mid \mathcal{F}_{i-1}^U\right] \right| \le C \sum_{i=1}^m \left| (\Delta_i^n U)^3\, \mathbb{E}\left[(\varepsilon(n,i) - m_1)^3 \,\middle|\, \mathcal{F}_{i-1}^U \right] \right| n^{-\frac{3}{2}} \xrightarrow{p} 0.$$

Consequently,

$$\sum_{i=1}^m \left(\mathbb{E}\left[(\xi_i^n)^2 \,\middle|\, \mathcal{F}_{i-1}^U \right] - \left(\mathbb{E}\left[\xi_i^n \mid \mathcal{F}_{i-1}^U\right] \right)^2 \right) \xrightarrow{p} a_1^2 \left(m_2 - m_1^2\right) \int_0^t (u_s v_s)^2\, ds.$$

Finally, by the analogous arguments as in the above proofs it can be easily seen that $\sum_{i=1}^m \mathbb{E}\left[(\xi_i^n)^4 \,\middle|\, \mathcal{F}_{i-1}^U \right] \xrightarrow{p} 0$, which implies

$$\sum_{i=1}^m \xi_i^n \xrightarrow{weakly} a_1 \sqrt{m_2 - m_1^2} \int_0^t u_s v_s\, dW'_s + a_2 \left(m_2 - m_1^2\right) \int_0^t (u_s v_s)^2\, ds,$$

which completes the proof.

□

B Proof of Theorem 4.2

The proof of Theorem 4.2 follows from a series of lemmas and theorems presented below. Consider the decomposition

$$\sum_{i=1}^{M_n} \overline{Y}^n_{(i-1)\ell_n+1} \left(\overline{\Delta}^n_{(i-1)\ell_n+1} - \frac{1}{k_n} \sum_{j=0}^{k_n-1} \overline{\Delta}^n_{(i-1)\ell_n+1+j} \right) = \mathsf{D}_1 + \mathsf{D}_2 + \mathsf{D}_3 + \mathsf{D}_4,$$

where

$$\begin{aligned}
\mathsf{D}_1 &= \sum_{i=1}^{M_n} \sigma_{t(n,(i-1)\ell_n)} \overline{W}^n_{(i-1)\ell_n+1} \left(\overline{\Delta}^n_{(i-1)\ell_n+1} - \mathbb{E}\left[\overline{\Delta}^n_{(i-1)\ell_n+1} \,\middle|\, \mathcal{F}_{(i-1)\ell_n} \right] \right), \\
\mathsf{D}_2 &= \sum_{i=1}^{M_n} \overline{U}^n_{(i-1)\ell_n+1} \left(\overline{\Delta}^n_{(i-1)\ell_n+1} - \mathbb{E}\left[\overline{\Delta}^n_{(i-1)\ell_n+1} \,\middle|\, \mathcal{F}_{(i-1)\ell_n} \right] \right), \\
\mathsf{D}_3 &= \sum_{i=1}^{M_n} \left(\overline{X}^n_{(i-1)\ell_n+1} - \sigma_{t(n,(i-1)\ell_n)} \overline{W}^n_{(i-1)\ell_n+1} \right) \left(\overline{\Delta}^n_{(i-1)\ell_n+1} - \mathbb{E}\left[\overline{\Delta}^n_{(i-1)\ell_n+1} \,\middle|\, \mathcal{F}_{(i-1)\ell_n} \right] \right), \\
\mathsf{D}_4 &= \sum_{i=1}^{M_n} \overline{Y}^n_{(i-1)\ell_n+1} \left(\mathbb{E}\left[\overline{\Delta}^n_{(i-1)\ell_n+1} \,\middle|\, \mathcal{F}_{(i-1)\ell_n} \right] - \frac{1}{k_n} \sum_{j=0}^{k_n-1} \overline{\Delta}^n_{(i-1)\ell_n+1+j} \right).
\end{aligned}$$

The leading term is D_1. Below we show that $n^{\lambda}\,\mathsf{D}_1$ converges in distribution to a centred normal random variable when $\lambda = \frac{3-2\,\delta}{4}$ (Theorem B.1) and the other terms D_2, D_3 and D_4 are asymptotically negligible (Theorems B.2, B.3 and B.4).

Lemma 1. *Under the assumptions of Theorem 4.2, for some constant $C > 0$ and for every $q > 0$, we have the following estimates*

$$\mathbb{E}\left[\left| \overline{Y}^n_{(i-1)\ell_n+1} \right|^q \,\middle|\, \mathcal{F}_{(i-1)\ell_n} \right] \leq C \left(\frac{\ell_n}{n} \right)^{q/2}.$$

and

$$\mathbb{E}\left[\left| \overline{X}^n_{(i-1)\ell_n+1} - \sigma_{t(n,(i-1)\ell_n)} \overline{W}^n_{(i-1)\ell_n+1} \right|^q \,\middle|\, \mathcal{F}_{(i-1)\ell_n} \right] \leq C \left(\frac{\ell_n}{n} \right)^{q}.$$

Proof. The proof of the first inequality is analogous to the proof of Lemma 1 of Podolskij and Vetter (2009). □

Lemma 2. *Let H be any bounded cadlag process. Then, under the assumptions of Theorem 4.2, as $n \to \infty$, we have:*

$$\sum_{i=1}^{M_n} H_{t(n,(i-1)\ell_n)} \frac{\ell_n}{n} \overset{u.c.p.}{\longrightarrow} \int_0^t \frac{H_s}{m_1 v_s}\, ds.$$

Proof. Using the results in Lemma 2.3(b) of Hayashi et al. (2011) we have

$$\sum_{i=1}^{m} H_{t(n,i)}\, \Delta\left(n, i\right) = \sum_{i=1}^{m} H_{t(n,i)} \frac{1}{n} v^n_{t(n,i-1)}\, \varepsilon\left(n, i\right) \overset{u.c.p.}{\longrightarrow} \int_0^t H_s\, ds.$$

Similar computations show that

$$\sum_{i=1}^{M_n} \left(H_{t(n,(i-1)\ell_n)} \sum_{j=1}^{\ell_n} \Delta(n, (i-1)\ell_n + j) \right) \overset{u.c.p.}{\longrightarrow} \int_0^t H_s\, ds.$$

Consider now the following decomposition

$$\sum_{i=1}^{M_n}\left(H_{t(n,(i-1)\ell_n)}\sum_{j=1}^{\ell_n}\Delta(n,(i-1)\ell_n+j)\right)=\mathsf{E}_1-\mathsf{E}_2-\mathsf{E}_3,$$

where

$$\begin{aligned}
\mathsf{E}_1 &= \sum_{i=1}^{M_n}H_{t(n,(i-1)\ell_n)}\frac{\ell_n}{n}m_1 v_{t(n,(i-1)\ell_n)},\\
\mathsf{E}_2 &= \sum_{i=1}^{M_n}H_{t(n,(i-1)\ell_n)}\left(\frac{\ell_n}{n}m_1 v_{t(n,(i-1)\ell_n)}-\mathbb{E}\left[\sum_{j=1}^{\ell_n}\Delta(n,(i-1)\ell_n+j)\,\middle|\,\mathcal{F}_{(i-1)\ell_n}\right]\right),\\
\mathsf{E}_3 &= \sum_{i=1}^{M_n}H_{t(n,(i-1)\ell_n)}\left(\mathbb{E}\left[\sum_{j=1}^{\ell_n}\Delta(n,(i-1)\ell_n+j)\,\middle|\,\mathcal{F}_{(i-1)\ell_n}\right]-\sum_{j=1}^{\ell_n}\Delta(n,(i-1)\ell_n+j)\right).
\end{aligned}$$

It can be easily seen that E_2 and E_3 are asymptotically negligible. Consequently,

$$\mathsf{E}_1\overset{u.c.p.}{\longrightarrow}\int_0^t H_s\,ds,$$

which implies the thesis

$$\sum_{i=1}^{M_n}H_{t(n,(i-1)\ell_n)}\frac{\ell_n}{n}\overset{u.c.p.}{\longrightarrow}\int_0^t\frac{H_s}{m_1 v_s}\,ds.$$

□

Theorem B.1. *Under the assumptions of Theorem 4.2 with* $\lambda=(3-2\delta)/4$ *we have, as* $n\to\infty$, *that*

$$n^\lambda\,\mathsf{D}_1\overset{weakly}{\longrightarrow}\sqrt{\Sigma_t^{\overline{\mathsf{B}}}}\mathcal{N}(0,1),$$

where $\mathsf{D}_1=\sum_{i=1}^{M_n}\sigma_{t(n,(i-1)\ell_n)}\overline{W}^n_{(i-1)\ell_n+1}\left(\overline{\Delta}^n_{(i-1)\ell_n+1}-\mathbb{E}\left[\overline{\Delta}^n_{(i-1)\ell_n+1}\,\middle|\,\mathcal{F}_{(i-1)\ell_n}\right]\right)$ *and*

$$\Sigma_t^{\overline{\mathsf{B}}}=\vartheta\int_0^t\sigma_s^2\,v_s^3\,\psi_{1,2}\,ds, \tag{35}$$

where

$$\psi_{1,2}=\int_0^1\int_0^1(g_u)^2(g_s)^2K(u,s)\,du\,ds, \tag{36}$$

with

$$K(u,s)=\begin{cases}\mathbb{E}\left[\varepsilon(n,1)\left(\varepsilon(n,1)-m_1\right)^2\right], & \text{if } u=s,\\ \mathbb{E}\left[\varepsilon(n,2)\left(\varepsilon(n,1)-m_1\right)^2\right], & \text{if } u\neq s.\end{cases}$$

Moreover, we have the joint convergence:

$$\left(n^\lambda\,\mathsf{D}_1,n^{2\lambda}\,\Sigma_t^{(\overline{\mathsf{B}},n)}\right)\overset{weakly}{\longrightarrow}\left(\sqrt{\Sigma_t^{\overline{\mathsf{B}}}}\mathcal{N}(0,1),\Sigma_t^{\overline{\mathsf{B}}}\right),$$

where

$$\Sigma_t^{(\overline{\mathsf{B}},n)}=\sum_{i=1}^{M_n}\left(\sigma_{t(n,(i-1)\ell_n)}\overline{W}^n_{(i-1)\ell_n+1}\left(\overline{\Delta}^n_{(i-1)\ell_n+1}-\mathbb{E}\left[\overline{\Delta}^n_{(i-1)\ell_n+1}\,\middle|\,\mathcal{F}_{(i-1)\ell_n}\right]\right)\right)^2 \tag{37}$$

Proof. We have

$$n^{\lambda}\,\mathsf{D}_1=\sum_{i=1}^{M_n}\xi_i^n,\quad n^{2\lambda}\,\Sigma_t^{(\overline{\mathsf{B}},n)}=\sum_{i=1}^{M_n}\left(\xi_i^n\right)^2,$$

where $\xi_i^n=n^{\lambda}\,\sigma_{t^n_{(i-1)\ell_n}}\,\overline{W}^n_{(i-1)\ell_n+1}\left(\overline{\Delta}^n_{(i-1)\ell_n+1}-\mathbb{E}\left[\overline{\Delta}^n_{(i-1)\ell_n+1}\,\middle|\,\mathcal{F}_{(i-1)\ell_n}\right]\right)$. In what follows, for notational convenience, we use the convention $\varepsilon_j^n=\varepsilon\left(n,j\right)$ and $t_j^n=t\left(n,j\right)$. Observe that under $\mathcal{H}_0$, $\mathbb{E}\left[\xi_i^n\,\middle|\,\mathcal{F}_{(i-1)\ell_n}\right]=0$. By conditioning on $\mathcal{F}'_{(i-1)\ell_n}=\mathcal{F}_{(i-1)\ell_n}\bigvee\sigma\left\{\Delta(n,i);0\le i\le N_t^n\right\}$ and using the law of iterated expectations we obtain

$$\begin{aligned}
&n^{-2\lambda}\,\sigma^{-2}_{t^n_{(i-1)\ell_n}}\,\mathbb{E}\left[\left(\xi_i^n\right)^2\,\middle|\,\mathcal{F}_{(i-1)\ell_n}\right]\\
=\;&\mathbb{E}\left[\mathbb{E}\left[\left(\overline{W}^n_{(i-1)\ell_n+1}\right)^2\,\middle|\,\mathcal{F}'_{(i-1)\ell_n}\right]\left(\overline{\Delta}^n_{(i-1)\ell_n+1}-\mathbb{E}\left[\overline{\Delta}^n_{(i-1)\ell_n+1}\,\middle|\,\mathcal{F}_{(i-1)\ell_n}\right]\right)^2\,\middle|\,\mathcal{F}_{(i-1)\ell_n}\right]\\
=\;&\mathbb{E}\left[\left(\sum_{j=1}^{\ell_n}\left(g_j^n\right)^2\Delta(n,(i-1)\ell_n+1+j)\right)\left(\overline{\Delta}^n_{(i-1)\ell_n+1}-\mathbb{E}\left[\overline{\Delta}^n_{(i-1)\ell_n+1}\,\middle|\,\mathcal{F}_{(i-1)\ell_n}\right]\right)^2\,\middle|\,\mathcal{F}_{(i-1)\ell_n}\right]\\
=\;&n^{-2}\,\mathbb{E}_{(i-1)\,\ell_n}\left[\left(\sum_{j=1}^{\ell_n}\left(g_j^n\right)^2 v^n_{t^n_{(i-1)\ell_n+j}}\,\varepsilon^n_{(i-1)\ell_n+1+j}\right)\left(\sum_{j=1}^{\ell_n}g_j^n\left(v^n_{t^n_{(i-1)\ell_n+j)}}\varepsilon^n_{(i-1)\ell_n+1+j}-m_1\mathbb{E}_{(i-1)\,\ell_n}\left[v^n_{t^n_{(i-1)\ell_n+j)}}\right]\right)\right)^2\right]\\
=\;&n^{-3}\sum_{j=1}^{\ell_n}\sum_{s=1}^{\ell_n}\left(g_j^n\right)^2\left(g_s^n\right)^2\mathbb{E}\left[v^n_{t^n_{(i-1)\ell_n+j}}\,\varepsilon^n_{(i-1)\ell_n+1+j}\left(v^n_{t^n_{(i-1)\ell_n+s}}\varepsilon^n_{(i-1)\ell_n+1+s}-m_1\mathbb{E}\left[v^n_{t^n_{(i-1)\ell_n+s}}\,\middle|\,\mathcal{F}_{(i-1)\ell_n}\right]\right)^2\,\middle|\,\mathcal{F}_{(i-1)\ell_n}\right].
\end{aligned}$$

Hence we can write

$$\mathbb{E}\left[\left(\xi_i^n\right)^2\,\middle|\,\mathcal{F}_{(i-1)\ell_n}\right]=n^{2\lambda-3}\,\ell_n^2\sigma^2_{t^n_{(i-1)\ell_n}}\,\psi^n_{1,2}\left(i\right)$$

where

$$\psi^n_{1,2}\left(i\right)=\ell_n^{-2}\sum_{j,s}\left(g_j^n\right)^2\left(g_s^n\right)^2\mathbb{E}\left[v^n_{t^n_{(i-1)\ell_n+j}}\,\varepsilon^n_{(i-1)\ell_n+1+j}\left(v^n_{t^n_{(i-1)\ell_n+s}}\varepsilon^n_{(i-1)\ell_n+1+s}-m_1\mathbb{E}\left[v^n_{t^n_{(i-1)\ell_n+s}}\,\middle|\,\mathcal{F}_{(i-1)\ell_n}\right]\right)^2\,\middle|\,\mathcal{F}_{(i-1)\ell_n}\right],$$

Note that, as $n\to\infty$,

$$\psi^n_{1,2}\left(i\right)-v^n_{t^n_{(i-1)\ell_n+j}}\left(v^n_{t(n,(i-1)\ell_n+s))}\right)^2\psi_{1,2}\xrightarrow{p}0,$$

where $\psi_{1,2}$ is defined in equation (36). On the other hand, since $\ell_n=\vartheta n^{\frac{1}{2}+\delta}$ and $\lambda=\frac{3-2\delta}{4}$, we have:

$$n^{2\lambda-3}\ell_n^2=\vartheta^2 n^{2\lambda-2-2\delta}=\vartheta\frac{\ell_n}{n}.$$

Consequently, by Lemma 2,

$$\sum_{i=1}^{M_n}\mathbb{E}\left[\left(\xi_i^n\right)^2\,\middle|\,\mathcal{F}_{(i-1)\ell_n}\right]\xrightarrow{p}\vartheta\int_0^t\sigma_s^2\,v_s^3\,\psi_{1,2}\,ds.$$

Tedious but straightforward computations allow to show that

$$\sum_{i=1}^{M_n}\mathbb{E}\left[\left(\xi_i^n\right)^4\,\middle|\,\mathcal{F}_{(i-1)\ell_n}\right]\xrightarrow{u.c.p.}0,$$

which completes the proof of the convergence $n^{\lambda}\,\mathsf{D}_1\overset{weakly}{\longrightarrow}\sqrt{\Sigma_t^{\overline{\mathsf{B}}}}\mathcal{N}\left(0,1\right)$.

□

Theorem B.2. *Under the assumptions of Theorem 4.2 with* $\lambda = \frac{3-2\delta}{4}$ *we have, as* $n \to \infty$*, that*

$$n^{\lambda}\, \mathsf{D}_2 \xrightarrow{u.c.p.} 0.$$

Proof of Theorem B.2. Let

$$\begin{aligned}
\overline{d}^{n}_{(i-1)\ell_n+1} &= \overline{\Delta}^{n}_{(i-1)\ell_n+1} - \mathbb{E}\left[\overline{\Delta}^{n}_{(i-1)\ell_n+1} \,\middle|\, \mathcal{F}_{(i-1)\ell_n}\right] \\
&= \sum_{j=1}^{\ell_n-1} g^n_j \left(\Delta\left(n,(i-1)\ \ell_n+1+j\right) - \mathbb{E}\left[\Delta\left(n,(i-1)\ \ell_n+1+j\right) \,\middle|\, \mathcal{F}_{(i-1)\ell_n}\right]\right) \\
&= \frac{1}{n}\sum_{j=1}^{\ell_n-1} g^n_j \left(v^n_{t(n,(i-1)\,\ell_n+j)}\, \varepsilon\left(n,(i-1)\ \ell_n+1+j\right) - m_1\, \mathbb{E}\left[v^n_{t(n,(i-1)\,\ell_n+1+j)} \,\middle|\, \mathcal{F}_{(i-1)\ell_n}\right]\right).
\end{aligned}$$

The statement we want to prove is now equivalent to

$$\sum_{i=1}^{M_n} \mathbb{E}\left[n^{\lambda}\left|\overline{U}^{n}_{(i-1)\ell_n+1}\overline{d}^{n}_{(i-1)\ell_n+1}\right|\right] \longrightarrow 0.$$

To prove it, consider first that, by Hōlder inequality, we have

$$\mathbb{E}\left[n^{\lambda}\left|\overline{U}^{n}_{(i-1)\ell_n+1}\overline{d}^{n}_{(i-1)\ell_n+1}\right|\right] \le n^{\lambda}\sqrt{\mathbb{E}\left[\left|\overline{U}^{n}_{(i-1)\ell_n+1}\right|^2\right]\mathbb{E}\left[\left|\overline{d}^{n}_{(i-1)\ell_n+1}\right|^2\right]}. \tag{38}$$

Notice that, for every index i, we have:

$$\overline{U}^n_i = \sum_{j=1}^{\ell_n-1}\left(g^n_j - g^n_{j+1}\right) U_{t(n,i+j)}.$$

Consequently,

$$\mathbb{E}\left[\left|\overline{U}^{n}_{(i-1)\ell_n+1}\right|^2\right] = \omega^2 \sum_{j=1}^{\ell_n-1}\left(g^n_j - g^n_{j+1}\right)^2 = \omega^2 \frac{1}{\ell_n}\psi^n_1, \tag{39}$$

where $\omega^2 = \mathbb{E}\left[U^2_{t(n,i)}\right]$ is the variance of the noise term and

$$\psi^n_1 = \sum_{j=1}^{\ell_n-1}\left(\frac{g^n_{j+1}-g^n_j}{\frac{1}{\ell_n}}\right)^2 \frac{1}{\ell_n} \to \int_0^1 \left(g'(s)\right)^2\, ds.$$

For the second term we have

$$\begin{aligned}
\mathbb{E}\left[\left|\overline{d}^{n}_{(i-1)\ell_n+1}\right|^2 \,\middle|\, \mathcal{F}_{(i-1)\ell_n}\right] &= \frac{1}{n^2}\sum_{j=0}^{\ell_n}\left(g^n_j\right)^2\left(m_2\mathbb{E}\left[\left(v^n_{t(n,(i-1)\ell_n+j)}\right)^2 \,\middle|\, \mathcal{F}_{(i-1)\ell_n}\right] - m_1^2\left(\mathbb{E}\left[v^n_{t(n,(i-1)\ell_n+j)} \,\middle|\, \mathcal{F}_{(i-1)\ell_n}\right]\right)^2\right) \\
&\le \frac{\ell_n}{n^2} C \frac{1}{\ell_n}\sum_{j=0}^{\ell_n}\left(g^n_j\right)^2 = \frac{\ell_n}{n^2} C \psi^n_2,
\end{aligned} \tag{40}$$

where $\psi^n_2 = \frac{1}{\ell_n}\sum_{j=0}^{\ell_n}\left(g^n_j\right)^2$ converges to $\psi_2 = \int_0^t \left(g(s)\right)^2 ds$. Combining (38), (39) and (40) we obtain

$$\sum_{i=1}^{M_n} \mathbb{E}\left[n^{\lambda}\left|\overline{U}^{n}_{(i-1)\ell_n+1}\overline{d}^{n}_{(i-1)\ell_n+1}\right|\right] \le n^{\lambda-1}\sum_{i=1}^{M_n}\sqrt{\omega^2\psi^n_1 C_v \psi^n_2}.$$

Since $\ell_n = \vartheta n^{\frac{1}{2}+\delta}$ and $\lambda = \frac{3-2\delta}{4}$, we have

$$n^{\lambda-1} = \frac{\ell_n}{n}\vartheta^{-1} n^{\lambda-(\frac{1}{2}+\delta)} = \frac{\ell_n}{n}\vartheta^{-1} n^{\frac{1-6\delta}{4}}.$$

Consequently, using Lemma 2, we obtain

$$\sum_{i=1}^{M_n} \sqrt{\omega^2 \psi_1^n C_v \psi_2^n}\, n^{\lambda-1} = n^{\frac{1-6\delta}{4}} \vartheta^{-1} \sum_{i=1}^{M_n} \sqrt{\omega^2 \psi_1^n C_v \psi_2^n} \frac{\ell_n}{n} \longrightarrow 0,$$

which implies that $\sum_{i=1}^{M_n} \mathbb{E}\left[n^{\lambda} \left| \overline{U}^n_{(i-1)\ell_n+1} \overline{d}^n_{(i-1)\ell_n+1} \right| \right] \longrightarrow 0$ and completes the proof.

□

Theorem B.3. *Under the assumptions of Theorem 4.2 hold and with $\lambda = \frac{3-2\delta}{4}$ we have that, as $n \to \infty$, the following convergence holds*

$$n^{\lambda}\, \mathsf{D}_3 \overset{u.c.p.}{\longrightarrow} 0.$$

Proof. We have

$$n^{\lambda}\, \mathsf{D}_3 = \sum_{i=1}^{M_n} n^{\lambda} \left(\overline{X}^n_{(i-1)\ell_n+1} - \sigma_{t(n,(i-1)\ell_n)} \overline{W}^n_{(i-1)\ell_n+1} \right) \overline{d}^n_{(i-1)\ell_n+1},$$

where $\overline{d}^n_{(i-1)\ell_n+1}$ is defined as in Theorem B.2. Then, it is enough to show that

$$\sum_{i=1}^{M_n} \mathbb{E}\left[n^{\lambda} \left| \left(\overline{X}^n_{(i-1)\ell_n+1} - \sigma_{t(n,(i-1)\ell_n)} \overline{W}^n_{(i-1)\ell_n+1} \right) \overline{d}^n_{(i-1)\ell_n+1} \right| \,\middle|\, \mathcal{F}_{(i-1)\ell_n} \right] \overset{p}{\longrightarrow} 0.$$

By Lemma 1, we have

$$\mathbb{E}\left[\left(\overline{X}^n_{(i-1)\ell_n+1} - \sigma_{t(n,(i-1)\ell_n)} \overline{W}^n_{(i-1)\ell_n+1} \right)^2 \,\middle|\, \mathcal{F}_{(i-1)\ell_n} \right] \leq \left(\frac{\ell_n}{n} \right)^2.$$

Additionally, inequality (40) from the proof of Theorem B.2 gives

$$\mathbb{E}\left[\left| \overline{d}^n_{(i-1)\ell_n+1} \right|^2 \,\middle|\, \mathcal{F}_{(i-1)\ell_n} \right] \leq C\, \frac{\ell_n}{n^2}.$$

Consequently, using Hōlder inequality and the above estimates we obtain

$$\sum_{i=1}^{M_n} \mathbb{E}\left[n^{\lambda} \left| \left(\overline{X}^n_{(i-1)\ell_n+1} - \sigma_{t(n,(i-1)\ell_n)} \overline{W}^n_{(i-1)\ell_n+1} \right) \overline{d}^n_{(i-1)\ell_n+1} \right| \,\middle|\, \mathcal{F}_{(i-1)\ell_n} \right] \leq C \sum_{i=1}^{M_n} n^{\lambda} \frac{\ell_n}{n} \left(\frac{\ell_n}{n^2} \right)^{1/2} = O\left(n^{\frac{2\delta-1}{4}} \right) \longrightarrow 0,$$

which completes the proof. □

Theorem B.4. *Under the assumptions of Theorem 4.2 hold and with $\lambda = \frac{3-2\delta}{4}$ we have that, as $n \to \infty$, the following convergence holds*

$$n^{\lambda}\, \mathsf{D}_4 \overset{u.c.p.}{\longrightarrow} 0.$$

Proof. Consider the decomposition

$$n^{\lambda}\, \mathsf{D}_4 = \sum_{i=1}^{M_n} \varphi^n_{(i-1)\ell_n+1} + \sum_{i=1}^{M_n} \eta^n_{(i-1)\ell_n+1},$$

where

$$\varphi^n_{(i-1)\ell_n+1} = n^\lambda \overline{Y}^n_{(i-1)\ell_n+1}\left(\frac{1}{k_n}\sum_{k=0}^{k_n-1}\left(\mathbb{E}\left[\overline{\Delta}^n_{(i-1)\ell_n+1}\;\middle|\;\mathcal{F}_{(i-1)\ell_n}\right]-\mathbb{E}\left[\overline{\Delta}^n_{(i-1)\ell_n+1+k}\;\middle|\;\mathcal{F}_{(i-1)\ell_n}\right]\right)\right),$$

$$\eta^n_{(i-1)\ell_n+1} = n^\lambda \overline{Y}^n_{(i-1)\ell_n+1}\left(\frac{1}{k_n}\sum_{k=0}^{k_n-1}\left(\mathbb{E}\left[\overline{\Delta}^n_{(i-1)\ell_n+1+k}\;\middle|\;\mathcal{F}_{(i-1)\ell_n}\right]-\overline{\Delta}^n_{(i-1)\ell_n+1+k}\right)\right).$$

It is enough to prove that the terms $\sum_{i=1}^{M_n}\varphi^n_{(i-1)\ell_n+1}$ and $\sum_{i=1}^{M_n}\eta^n_{(i-1)\ell_n+1}$ are asymptotically negligible. By Lemma 1, we have

$$\mathbb{E}\left[\left|\varphi^n_{(i-1)\ell_n+1}\right|\;\middle|\;\mathcal{F}_{(i-1)\ell_n}\right]\le n^\lambda\left(\frac{\ell_n}{n}\right)^{1/2}\frac{1}{k_n}\sum_{k=0}^{k_n-1}\left|\mathbb{E}\left[\overline{\Delta}^n_{(i-1)\ell_n+1}-\overline{\Delta}^n_{(i-1)\ell_n+1+k}\;\middle|\;\mathcal{F}_{(i-1)\ell_n}\right]\right|.$$

Since v^n_t is a semimartingale we obtain:

$$\begin{aligned}\left|\mathbb{E}\left[\overline{\Delta}^n_{(i-1)\ell_n+1}-\overline{\Delta}^n_{(i-1)\ell_n+1+k}\;\middle|\;\mathcal{F}_{(i-1)\ell_n}\right]\right| &= \left|\frac{1}{n}\sum_{j=1}^{\ell_n}g^n_j m_1\mathbb{E}\left[v^n_{t(n,(i-1)\ell_n+j)}-v^n_{t(n,(i-1)\ell_n+k+j)}\;\middle|\;\mathcal{F}_{(i-1)\ell_n}\right]\right|\\ &\le \frac{1}{n}\sum_{j=1}^{\ell_n}g^n_j m_1\left|\mathbb{E}\left[v^n_{t(n,(i-1)\ell_n+j)}-v^n_{t(n,(i-1)\ell_n+k+j)}\;\middle|\;\mathcal{F}_{(i-1)\ell_n}\right]\right|\\ &\le C\,\frac{1}{n}\sum_{j=1}^{\ell_n}g^n_j\frac{k}{n}=C\,\frac{k\ell_n}{n^2}\widetilde{\psi}^n_1,\end{aligned}$$

where $\widetilde{\psi}^n_1=\frac{1}{\ell_n}\sum_{j=1}^n g^n_j$. Consequently,

$$\mathbb{E}\left[\left|\varphi^n_{(i-1)\ell_n+1}\right|\;\middle|\;\mathcal{F}_{(i-1)\ell_n}\right]\le C\,n^\lambda\left(\frac{\ell_n}{n}\right)^{1/2}\frac{1}{k_n}\sum_{k=0}^{k_n-1}\frac{k\ell_n}{n^2}\widetilde{\psi}^n_1=C\,\widetilde{\psi}^n_1\,n^\lambda\left(\frac{\ell_n}{n}\right)^{1/2}\frac{\ell_n(k_n-1)}{n^2}.$$

Since $\ell_n\sim n^{\frac12+\delta}$ and $\lambda=\frac{3-2\delta}{4}$ and $k_n\sim n^\nu$, we have

$$n^\lambda\left(\frac{\ell_n}{n}\right)^{1/2}\frac{k_n-1}{n}\sim n^{\nu-\frac12}\to 0.$$

Hence, using Lemma 2, we obtain:

$$\sum_{i=1}^{M_n}\mathbb{E}\left[\left|\varphi^n_{(i-1)\ell_n+1}\right|\;\middle|\;\mathcal{F}_{(i-1)\ell_n}\right]\le n^\lambda\left(\frac{\ell_n}{n}\right)^{1/2}\frac{k_n-1}{n}\sum_{i=1}^{M_n}C\,\widetilde{\psi}^n_1\,\frac{\ell_n}{n}\xrightarrow{p}0,$$

which implies that the array $\sum_{i=1}^{M_n}\varphi^n_{(i-1)\ell_n+1}$ is asymptotically negligible. Next, using Hōlder inequality and Lemma 1, we obtain

$$\mathbb{E}\left[\left|\eta^n_{(i-1)\ell_n+1}\right|\;\middle|\;\mathcal{F}_{(i-1)\ell_n}\right]\le n^\lambda\left(\frac{\ell_n}{n}\right)^{1/2}\sqrt{\mathbb{E}\left[\left(\frac{1}{k_n}\sum_{k=0}^{k_n-1}\left[\mathbb{E}\left[\overline{\Delta}^n_{(i-1)\ell_n+1+k}\;\middle|\;\mathcal{F}_{(i-1)\ell_n}\right]-\overline{\Delta}^n_{(i-1)\ell_n+1+k}\right]\right)^2\;\middle|\;\mathcal{F}_{(i-1)\ell_n}\right]}.$$

Notice that for every $s>k$, we have

$$\mathbb{E}\left[\left(\mathbb{E}\left[\overline{\Delta}^n_{(i-1)\ell_n+1+k}\;\middle|\;\mathcal{F}_{(i-1)\ell_n}\right]-\overline{\Delta}^n_{(i-1)\ell_n+1+k}\right)\left(\mathbb{E}\left[\overline{\Delta}^n_{(i-1)\ell_n+1+s}\;\middle|\;\mathcal{F}_{(i-1)\ell_n}\right]-\overline{\Delta}^n_{(i-1)\ell_n+1+s}\right)\;\middle|\;\mathcal{F}_{(i-1)\ell_n}\right]=0.$$

Consequently,

$$\mathbb{E}\left[\left|\eta^n_{(i-1)\ell_n+1}\right| \;\middle|\; \mathcal{F}_{(i-1)\ell_n}\right] \le n^{\lambda}\left(\frac{\ell_n}{n}\right)^{1/2}\frac{1}{k_n}\sqrt{\sum_{k=0}^{k_n-1}\mathbb{E}\left[\left(\overline{\Delta}^n_{(i-1)\ell_n+1+k}-\mathbb{E}\left[\overline{\Delta}^n_{(i-1)\ell_n+1+k}\;\middle|\;\mathcal{F}_{(i-1)\ell_n}\right]\right)^2\;\middle|\;\mathcal{F}_{(i-1)\ell_n}\right]}.$$

For every $k = 0, 1, \dots, k_n - 1$ and indicating with $q = (i-1)\ell_n$ we have

$$\overline{\Delta}^n_{q+1+k} - \mathbb{E}\left[\overline{\Delta}^n_{q+1+k}\;\middle|\;\mathcal{F}_q\right] = \frac{1}{n}\sum_{j=1}^{\ell_n} g^n_j\left(v^n_{t(n,q+k+j)}\varepsilon(n,q+k+j+1) - \mathbb{E}\left[v^n_{t(n,q+k+j)}\;\middle|\;\mathcal{F}_q\right]m_1\right).$$

Consequently,

$$\begin{aligned}\mathbb{E}\left[\left(\overline{\Delta}^n_{q+1+k}-\mathbb{E}\left[\overline{\Delta}^n_{q+1+k}\;\middle|\;\mathcal{F}_q\right]\right)^2\;\middle|\;\mathcal{F}_q\right] &= \frac{1}{n^2}\sum_{j=1}^{\ell_n}\left(g^n_j\right)^2\left(\mathbb{E}\left[\left(v^n_{t(n,q+k+j)}\right)^2\;\middle|\;\mathcal{F}_q\right]m_2 - \left(\mathbb{E}\left[v^n_{t(n,q+k+j)}\;\middle|\;\mathcal{F}_q\right]m_1\right)^2\right)\\ &\le C\,\frac{\ell_n}{n^2}\widetilde{\psi}^n_2,\end{aligned}$$

where $\widetilde{\psi}^n_2 = \sum_{j=1}^{\ell_n}\left(g^n_j\right)^2\frac{1}{\ell_n}$. This implies that

$$\mathbb{E}\left[\left(\frac{1}{k_n}\sum_{k=1}^{k_n-1}\mathbb{E}\left[\overline{\Delta}^n_{(i-1)\ell_n+1+k}\;\middle|\;\mathcal{F}_{(i-1)\ell_n}\right]-\overline{\Delta}^n_{(i-1)\ell_n+1+k}\right)^2\;\middle|\;\mathcal{F}_{(i-1)\ell_n}\right] \le C\,\widetilde{\psi}^n_2\,\frac{\ell_n}{k_n n^2},$$

and

$$\mathbb{E}\left[\left|\eta^n_{(i-1)\ell_n+1}\right|\;\middle|\;\mathcal{F}_{(i-1)\ell_n}\right] \le C\sqrt{\widetilde{\psi}^n_2}\,n^{\lambda}\left(\frac{\ell_n}{n}\right)^{1/2}\frac{\ell_n^{1/2}}{n k_n^{1/2}}.$$

Since $\ell_n \sim n^{\frac{1}{2}+\delta}$ and $\lambda = \frac{3-2\delta}{4}$ and $k_n \sim n^{\nu}$, we have $n^{\lambda}n^{-1/2}k_n^{-1/2} \sim n^{\frac{1-2\delta-2\nu}{4}} \longrightarrow 0$. Hence, using Lemma 2, we obtain

$$\sum_{i=1}^{M_n}\mathbb{E}\left[\left|\eta^n_{(i-1)\ell_n+1}\right|\;\middle|\;\mathcal{F}_{(i-1)\ell_n}\right] \le n^{\lambda}n^{-1/2}k_n^{-1/2}\sum_{i=1}^{M_n}C\,\widetilde{\psi}^n_2\,\frac{\ell_n}{n} \xrightarrow{p} 0,$$

which implies that the array $\sum_{i=1}^{M_n}\eta^n_{(i-1)\ell_n+1}$ is asymptotically negligible and completes the proof.

□

Theorems B.1, B.2, B.3 and B.4 together imply that the numerator of the test statistic $\overline{\mathsf{B}}\left(Y,\{t\left(n,i\right)\}\right)$ converges in distribution to a normal random variable with random variance $\Sigma^{\overline{\mathsf{B}}}_t$ defined in equation (35).

Analogously to the proofs of Theorems B.2, B.3 and B.4 it is possible to prove that the difference between the denominator of the test statistic $\overline{\mathsf{B}}\left(Y,\{t\left(n,i\right)\}\right)$ and the estimator $\Sigma^{(\overline{\mathsf{B}},n)}_t$ defined in equation (37) of Theorem B.1 is negligible, which concludes the proof of Theorem 4.2.

RECENT PUBLICATIONS BY *CEIS Tor Vergata*

Aggregation Trees
Riccardo Di Francesco
CEIS Research Paper, 546, December 2022

Food Security during the COVID-19 Pandemic: the Impact of a Rural Development Program and Neighbourhood Spillover Effect in the Solomon Islands
Leonardo Becchetti, Sara Mancini and Sara Savastano
CEIS Research Paper, 545, December 2022

Attitude Towards Financial Planning of Italian Households
Marianna Brunetti, Rocco Ciciretti, Monica Gentile, Nadia Linciano and Paola Soccorso
CEIS Research Paper, 544, December 2022

Climate Policies, Macroprudential Regulation, and the Welfare Cost of Business Cycles
Barbara Annicchiarico, Marco Carli and Francesca Diluiso
CEIS Research Paper, 543, October 2022

The Social Value of Health: Amenable Deaths and Estimated the Gap with the Life Expectancy Frontier
Leonardo Becchetti, Gianluigi Conzo and Giovanni Trovato
CEIS Research Paper, 542, August 2022

Staying Strong, But For How Long? Mental Health During COVID-19 in Italy
Francesca Marazzi, Andrea Piano Mortari, Federico Belotti, Giuseppe Carrà, Ciro Cattuto, Joanna Kopinska, Daniela Paolotti and Vincenzo Atella
CEIS Research Paper, 541, April 2022

Sensitivity of Profitability in Cointegration-Based Pairs Trading
Marianna Brunetti and Roberta De Luca
CEIS Research Paper, 540, April 2022

Theory of Evolutionary Spectra for Heteroskedasticity and Autocorrelation Robust Inference in Possibly Misspecified and Nonstationary Models
Alessandro Casini
CEIS Research Paper, 539, April 2022

Administrative Reforms and Urban Development: Lessons from Italian Unification
Giulio Cainelli, Carlo Ciccarelli and Roberto Ganau
CEIS Research Paper, 538, April 2022

Exposure to Climate Shocks, Poverty and Happiness: The ”Three Little Pigs” Effect
Leonardo Becchetti, Sara Mancini and Sara Savastano
CEIS Research Paper, 537, April 2022