On the axiomatic approach to sharing the revenues from broadcasting sports leagues

Bergantiños, Gustavo and Moreno-Ternero, Juan D.

Universidade de Vigo, Universidad Pablo de Olavide

28 September 2020

On the axiomatic approach to sharing the revenues from broadcasting sports leagues*

Gustavo Bergantiños†

Juan D. Moreno-Ternero‡

September 28, 2020

Abstract

We take the axiomatic approach to uncover the structure of the revenue-sharing problem from broadcasting sports leagues. Our starting point is to explore the implications of three basic axioms: *additivity*, *order preservation* and *weak upper bound*. We show that the combination of these axioms characterizes a large family of rules, which is made of compromises between the *uniform* rule and *concede-and-divide*, such as the one represented by the *equal-split* rule. The members of the family are fully ranked according to the Lorenz dominance criterion, and the structure of the family guarantees the existence of a majority voting equilibrium. Strengthening some of the previous axioms, or adding new ones, we provide additional characterizations within the family. Weakening some of those axioms, we also characterize several families encompassing the original one.

JEL numbers: *D63, C71, Z20.*

Keywords: *resource allocation, broadcasting, sport leagues, axioms, concede-and-divide.*

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*Financial support from the Spanish Ministry of Economics and Competitiveness, through the research projects ECO2017-82241-R and ECO2017-83069-P, and Xunta of Galicia through grant ED431B 2019/34 is gratefully acknowledged.

†ECOSOT, Universidade de Vigo.

‡Department of Economics, Universidad Pablo de Olavide

1 Introduction

In a recent paper (Bergantiños and Moreno-Ternero, 2020a), we have introduced a formal model to analyze the problem of sharing the revenues from broadcasting sports leagues among participating teams, based on the audiences they generate. In this paper, we uncover the structure of this stylized model further, thanks to the axiomatic approach.

We start considering three basic axioms: *additivity*, *order preservation* and *weak upper bound*. The first one says that awards are additive on audiences. The second one says that awards preserve the order of teams' audiences. The third one says that individual awards are bounded above by the overall revenues obtained from the whole tournament. The three axioms are satisfied by three rules that stand out as focal to solve this problem (Bergantiños and Moreno-Ternero, 2020c). They are the *uniform* rule, which shares equally among all participating teams the overall revenues obtained in the whole tournament, the *equal-split* rule, which splits the revenue generated from each game equally among the participating teams, and *concede-and-divide*, which concedes each team the revenues generated from its fan base and divides equally the residual.

Our first result shows that the combination of the three axioms mentioned above actually characterizes the family of rules compromising between the *uniform* rule and *concede-and-divide*, which actually has the *equal-split* rule as a focal member. Each rule within the family is simply defined by a certain convex combination of the *uniform* rule and *concede-and-divide*. We shall refer to this family as the UC-family of rules.

We also show that all rules within the family satisfy the so-called *single-crossing property*, which allows one to separate those teams who benefit from one rule or the other, depending on the rank of their aggregate audiences. This has important implications. On the one hand, the existence of a majority voting equilibrium (e.g., Gans and Smart, 1996). That is, if we allow teams to vote for any rule within the family, then there exists a rule that cannot be overturned by any other rule within the family through majority rule. On the other hand, the rules within the family yield outcomes that are fully ranked according to the Lorenz dominance criterion (e.g., Hemming and Keen, 1983). More precisely, for each problem, and each pair of rules within the family, the outcome suggested by the rule associated with a higher parameter dominates (in the sense of Lorenz) the outcome suggested by the other rule, which is equivalent to saying that the former will be more egalitarian than the latter (e.g., Dasgupta et al., 1973).

We then proceed to consider additional axioms to the structure supporting the UC-family of rules. We start showing that if we add *non-negativity* (no team receives negative awards), then only a specific part of the family survives; namely the rules that are actually convex combinations of the *uniform* rule and the *equal-split* rule, which we shall dub the UE-family of rules. More interestingly, we can dismiss the *weak upper bound* axiom to characterize such a family. To wit, we show that a rule satisfies *additivity*, *order preservation* and *non-negativity* if and only if it is a member of the UE-family of rules. This was actually an open question in Bergantiños and Moreno-Ternero (2020c).

It turns out that the other *half* of the UC-family of rules; namely, the rules that are actually convex combinations of the *equal-split* rule and *concede-and-divide*, dubbed here the EC-family of rules, can also be singled out. To do so, one simply needs to strengthen the *weak upper bound* axiom to *maximum aspirations*, which says that no team can receive an amount higher than its *claim* (i.e., the overall revenues obtained from all the games in which the team was involved). As a matter of fact, *order preservation* is not required in its full force for this characterization, and the cleanest result states that *additivity*, *equal treatment of equals* and *maximum aspirations* characterize the EC-family of rules. This is almost equivalent to the characterization in Bergantiños and Moreno-Ternero (2020b).[1]

We also provide additional characterization results for families encompassing the UC-family of rules, by weakening some of the original axioms considered for its characterization. More precisely, we characterize the rules satisfying *additivity*, *equal treatment of equals*, and either *weak upper bound* or *non-negativity*. We also characterize the rules satisfying *additivity* and *order preservation* and, finally, the rules satisfying *additivity* and *equal treatment of equals*. In all cases, we obtain linear (albeit not convex) combinations of the focal rules mentioned above.

The rest of the paper is organized as follows. We introduce the model, axioms and rules in Section 2. In Section 3, we provide the characterization result leading towards the UC-family of rules and then explore other properties of it. In Section 4, we obtain further characterizations for specific members of the family. In Section 5 we characterize more general families encompassing the UC-family of rules. Finally, we conclude in Section 6.

[1] Therein, we use a stronger notion than *equal treatment of equals* indicating that two teams with the same claims receive the same awards.

2 The model

We consider the model introduced by Bergantiños and Moreno-Ternero (2020a). Let N describe a finite set of teams. Its cardinality is denoted by n. We assume $n \geq 3$. For each pair of teams $i, j \in N$, we denote by a_{ij} the broadcasting audience (number of viewers) for the game played by i and j at i's stadium. We use the notational convention that $a_{ii} = 0$, for each $i \in N$. Let $A \in \mathcal{A}_{n\times n}$ denote the resulting matrix of broadcasting audiences generated in the whole tournament involving the teams within N.[2] Each matrix $A \in \mathcal{A}_{n\times n}$ with zero entries in the diagonal will thus represent a *problem* and we shall refer to the set of problems as $\mathcal{P}$.[3]

Let $\alpha_i(A)$ denote the total audience achieved by team i, i.e.,

$$\alpha_i(A) = \sum_{j\in N}(a_{ij} + a_{ji}).$$

Without loss of generality, we normalize the revenue generated from each viewer to 1 (to be interpreted as the "pay per view" fee). Thus, we sometimes refer to $\alpha_i(A)$ by the *claim* of team i. When no confusion arises, we write α_i instead of $\alpha_i(A)$. We define $\overline{\alpha}$ as the average audience of all teams. Namely,

$$\overline{\alpha} = \frac{\sum_{i\in N}\alpha_i}{n}.$$

For each $A \in \mathcal{A}_{n\times n}$, let $||A||$ denote the total audience of the tournament. Namely,

$$||A|| = \sum_{i,j\in N} a_{ij} = \frac{1}{2}\sum_{i\in N}\alpha_i = \frac{n\overline{\alpha}}{2}.$$

2.1 Rules

A (sharing) **rule** is a mapping that associates with each problem the list of the amounts the teams get from the total revenue. Without loss of generality, we normalize the revenue generated from each viewer to 1 (to be interpreted as the "pay per view" fee). Thus, formally, $R : \mathcal{P} \rightarrow \mathbb{R}^n$

[2]We are therefore assuming a round-robin tournament in which each team plays in turn against each other team twice: once home, another away. This is the usual format of the main European football leagues. Our model could also be extended to leagues in which some teams play other teams a different number of times and play-offs at the end of the regular season, which is the usual format of North American professional sports. In such a case, a_{ij} is the broadcasting audience in all games played by i and j at i's stadium.

[3]As the set N will be fixed throughout our analysis, we shall not explicitly consider it in the description of each problem.

is such that, for each $A \in \mathcal{P}$,

$$\sum_{i \in N} R_i(A) = ||A||.$$

The following three rules have been highlighted as focal for this problem (e.g., Bergantiños and Moreno-Ternero, 2020a; 2020b; 2020c).

The *uniform rule* divides equally among all teams the overall audience of the whole tournament. Formally,

Uniform, U: for each $A \in \mathcal{P}$, and each $i \in N$,

$$U_i(A) = \frac{||A||}{n} = \frac{\overline{\alpha}}{2}.$$

The *equal-split rule* divides the audience of each game equally, among the two participating teams. Formally,

Equal-split rule, ES: for each $A \in \mathcal{P}$, and each $i \in N$,

$$ES_i(A) = \frac{\alpha_i}{2}.$$

Concede-and-divide compares the performance of a team with the average performance of the other teams. Formally,

Concede-and-divide, CD: for each $A \in \mathcal{P}$, and each $i \in N$,

$$CD_i(A) = \alpha_i - \frac{\sum\limits_{j,k \in N \setminus \{i\}} (a_{jk} + a_{kj})}{n-2} = \frac{(n-1)\,\alpha_i - ||A||}{n-2} = \frac{2\,(n-1)\,\alpha_i - n\overline{\alpha}}{2(n-2)}.$$

The following family of rules encompasses the above three rules.

UC-family of rules $\left\{UC^{\lambda}\right\}_{\lambda \in [0,1]}$: for each $\lambda \in [0,1]$, each $A \in \mathcal{P}$, and each $i \in N$,

$$UC_i^{\lambda}(A) = (1-\lambda)U_i(A) + \lambda CD_i(A).$$

Equivalently,

$$UC_i^{\lambda}(A) = (1-\lambda)\frac{||A||}{n} + \lambda\frac{(n-1)\,\alpha_i - ||A||}{n-2} = \frac{\overline{\alpha}}{2} + \lambda\frac{n-1}{n-2}\left(\alpha_i - \overline{\alpha}\right). \tag{1}$$

At the risk of stressing the obvious, note that, when $\lambda = 0$, UC^{λ} coincides with the *uniform* rule, whereas, when $\lambda = 1$, UC^{λ} coincides with *concede-and-divide*. That is, $UC^0 \equiv U$ and $UC^1 \equiv CD$. Bergantiños and Moreno-Ternero (2020a) prove that for each $A \in \mathcal{P}$,

$$ES(A) = \frac{n}{2\,(n-1)}U(A) + \frac{n-2}{2\,(n-1)}CD(A).$$

That is, $UC^{\lambda} \equiv ES$, where $\lambda = \frac{n-2}{2(n-1)}$.[4]

Consequently, the *UC-family of rules* can be split in two.

On the one hand, the family of rules compromising between the uniform rule and the equal-split rule. Formally,

UE-family of rules $\{UE^{\beta}\}_{\beta\in[0,1]}$: for each $\beta \in [0,1]$, each $A \in \mathcal{P}$, and each $i \in N$,

$$UE_i^{\beta}(A) = (1-\beta)U_i(A) + \beta ES_i(A) = \frac{\overline{\alpha}}{2} + \frac{\beta}{2}(\alpha_i - \overline{\alpha}).$$

On the other hand, the family of rules compromising between the equal-split rule and concede-and-divide.[5] Formally,

EC-family of rules $\{EC^{\gamma}\}_{\gamma\in[0,1]}$: for each $\gamma \in [0,1]$, each $A \in \mathcal{P}$, and each $i \in N$,

$$EC_i^{\gamma}(A) = (1-\gamma)ES_i(A) + \gamma CD_i(A) = \frac{\alpha_i}{2} + \gamma\frac{n}{2(n-2)}(\alpha_i - \overline{\alpha}).$$

As Figure 1 illustrates, the family of UC rules is indeed the union of the family of UE rules and EC rules. Note that $UE^0 \equiv UC^0 \equiv U$, $EC^1 \equiv UC^1 \equiv CD$, whereas $ES \equiv UE^1 \equiv EC^0 \equiv UC^{\frac{n-2}{2(n-1)}}$ is the unique rule belonging to both families.

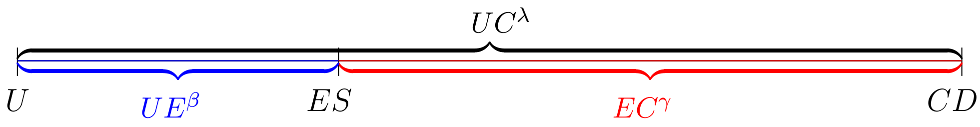

Figure 1. Illustration of the three families of rules. $UC^{\lambda} = UE^{\beta} \cup EC^{\gamma}$

2.2 Axioms

We now introduce the axioms we consider in this paper.

The first axiom we consider says that if two teams have the same audiences, then they should receive the same amount.

Equal treatment of equals: For each $A \in \mathcal{P}$, and each pair $i, j \in N$ such that $a_{ik} = a_{jk}$, and $a_{ki} = a_{kj}$, for each $k \in N \setminus \{i, j\}$,

$$R_i(A) = R_j(A).$$

[4]Note that λ approaches 0.5 (from below) for n arbitrarily large.

[5]We studied this family independently in Bergantiños and Moreno-Ternero (2020b).

The following axiom strengthens the previous one by saying that if the audience of team i is, game by game, not smaller than the audience of team j, then that team i should not receive less than team j.

Order preservation: For each $A \in \mathcal{P}$ and each pair $i, j \in N$, such that, for each $k \in N \backslash \{i, j\}$, $a_{ik} \geq a_{jk}$ and $a_{ki} \geq a_{kj}$ we have that

$$R_i(A) \geq R_j(A).$$

The next axiom says that each team should receive, at most, the total audience of the games played by the team.

Maximum aspirations: For each $A \in \mathcal{P}$ and each $i \in N$,

$$R_i(A) \leq \alpha_i.$$

Alternatively, one could consider a weaker upper bound with the total audience of all games in the tournament.

Weak upper bound: For each $A \in \mathcal{P}$ and each $i \in N$,

$$R_i(A) \leq ||A|| \,.$$

The next axiom provides instead a lower bound as it says that no team should receive negative awards. Formally,

Non-negativity. For each $A \in \mathcal{P}$ and $i \in N$,

$$R_i(A) \geq 0.$$

It is not difficult to show that both *maximum aspirations* and *non-negativity* imply *weak upper bound*.

The next axiom says that revenues should be additive on A. Formally,

Additivity: For each pair A and $A' \in \mathcal{P}$

$$R(A + A') = R(A) + R(A').$$

The final two axioms refer to the performance of the rule with respect to somewhat pathological teams. Null team says that if a team has a null audience, then such a team gets no revenue. Essential team says that if only the games played by some team have positive audience, then such a team should receive all its audience. Formally,

Null team: For each $A \in \mathcal{P}$, and each $i \in N$, such that $a_{ij} = 0 = a_{ji}$, for each $j \in N$,

$$R_i(A) = 0.$$

Essential team: For each $A \in \mathcal{P}$ and each $i \in N$ such that $a_{jk} = 0$ for each pair $\{j,k\} \in N \backslash \{i\}$,

$$R_i(A) = \alpha_i.$$

3 The benchmark family

We start this section with a characterization result of the UC-family of rules, our benchmark family.

Theorem 1 *A rule satisfies additivity, order preservation and weak upper bound if and only if it is a member of the UC-family of rules.*

Proof. It is not difficult to show that both the *uniform rule* and *concede-and-divide* satisfy all the axioms in the statement. It follows from there that all the members of the *UC-family of rules* satisfy them too.

Conversely, let R be a rule satisfying the three axioms. Note that, then, R satisfies *equal treatment of equals* too. Let $A \in \mathcal{P}$. For each pair $i, j \in N$, with $i \neq j$, let 1^{ij} denote the matrix with the following entries:

$$1^{ij}_{kl} = \begin{cases} 1 & \text{if } (k,l) = (i,j) \\ 0 & \text{otherwise.} \end{cases}$$

Notice that $1^{ij}_{ji} = 0$.

Let $k \in N$. By *additivity*,

$$R_k(A) = \sum_{i,j \in N : i \neq j} a_{ij} R_k\left(1^{ij}\right). \tag{2}$$

By *equal treatment of equals*, for each pair $k, l \in N \backslash \{i,j\}$ we have that $R_i\left(1^{ij}\right) = R_j\left(1^{ij}\right) = x^{ij}$, and $R_k\left(1^{ij}\right) = R_l\left(1^{ij}\right) = z^{ij}$. As $\sum_{k \in N} R_j\left(1^{ij}\right) = ||1^{ij}|| = 1$, we deduce that

$$z^{ij} = \frac{1 - 2x^{ij}}{n-2}.$$

Let $k \in N \setminus \{i,j\}$. By *additivity*, $R_j\left(1^{ij}+1^{ik}\right)=x^{ij}+z^{ik}$, and $R_k\left(1^{ij}+1^{ik}\right)=z^{ij}+x^{ik}$. By *equal treatment of equals*, $R_j\left(1^{ij}+1^{ik}\right)=R_k\left(1^{ij}+1^{ik}\right)$. Thus,

$$\begin{aligned} x^{ij}+\frac{1-2x^{ik}}{n-2} &= x^{ik}+\frac{1-2x^{ij}}{n-2} \Leftrightarrow \\ (n-2)\,x^{ij}+1-2x^{ik} &= (n-2)\,x^{ik}+1-2x^{ij} \Leftrightarrow \\ x^{ij} &= x^{ik} \end{aligned}$$

Therefore, there exists $x \in \mathbb{R}$ such that for each $\{i,j\} \subset N$,

$$\begin{aligned} R_i\left(1^{ij}\right) &= R_j\left(1^{ij}\right)=x, \text{ and} \\ R_l\left(1^{ij}\right) &= \frac{1-2x}{n-2} \text{ for each } l \in N \setminus \{i,j\}. \end{aligned}$$

By *weak upper bound*,

$$x=R_i\left(1^{ij}\right) \leq 1.$$

Let $k \in N \setminus \{i,j\}$. By *order preservation*,

$$x=R_i\left(1^{ij}\right) \geq R_k\left(1^{ij}\right)=\frac{1-2x}{n-2},$$

which implies that $x \geq \frac{1}{n}$.

Let

$$\lambda=\frac{nx-1}{n-1}.$$

As $\frac{1}{n} \leq x \leq 1$, it follows that $0 \leq \lambda \leq 1$.

Then,

$$UC_k^\lambda\left(1^{ij}\right)=(1-\lambda)U_k\left(1^{ij}\right)+\lambda CD_k\left(1^{ij}\right)=\begin{cases} (1-\lambda)\frac{1}{n}+\lambda=x & \text{if } k=i,j \\ (1-\lambda)\frac{1}{n}-\lambda\frac{1}{n-2}=\frac{1-2x}{n-2} & \text{otherwise.} \end{cases}$$

Thus, $UC^\lambda\left(1^{ij}\right)=R\left(1^{ij}\right)$. As UC^λ and R satisfy *additivity*, we deduce from here that $UC^\lambda(A)=R(A)$, for each $A \in \mathcal{P}$. ■

Theorem 1 shows that the UC-family comprises all rules satisfying three basic and intuitive properties. We show next that the family exhibits additional interesting features. To begin with, all rules within the family satisfy the so-called *single-crossing* property. That is, for each pair of rules within the family, and each problem $A \in \mathcal{P}$, there exists a team $i^* \in N$ separating those teams benefitting with one rule and those benefitting with the other. It turns out that i^* is precisely the team whose overall audience is closest (from below) to the average overall audience. Formally,

Proposition 1 *Let* $0 \leq \lambda_1 \leq \lambda_2 \leq 1$*, and* $A \in \mathcal{P}$ *such that, without loss of generality,* $N = \{1, \ldots, n\}$ *and* $\alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_n$*. Then, there exists* $i^* \in N$ *such that:*

(*i*) $UC_i^{\lambda_1}(A) \leq UC_i^{\lambda_2}(A)$ *for each* $i = 1, ..., i^*$ *and*

(*ii*) $UC_i^{\lambda_1}(A) \geq UC_i^{\lambda_2}(A)$ *for each* $i = i^* + 1, ..., n$*.*

Proof. Let $0 \leq \lambda_1 \leq \lambda_2 \leq 1$, and $A \in \mathcal{P}$ be such that $N = \{1, \ldots, n\}$ and $\alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_n$. Let $i \in N$. We distinguish two cases:

If $\alpha_i \leq \overline{\alpha}$, then

$$UC_i^{\lambda_1}(A) = \frac{\overline{\alpha}}{2} + \lambda_1 \frac{n-1}{n-2}(\alpha_i - \overline{\alpha}) \geq \frac{\overline{\alpha}}{2} + \lambda_2 \frac{n-1}{n-2}(\alpha_i - \overline{\alpha}) = UC_i^{\lambda_2}(A).$$

If $\alpha_i > \overline{\alpha}$, then

$$UC_i^{\lambda_1}(A) = \frac{\overline{\alpha}}{2} + \lambda_1 \frac{n-1}{n-2}(\alpha_i - \overline{\alpha}) \leq \frac{\overline{\alpha}}{2} + \lambda_2 \frac{n-1}{n-2}(\alpha_i - \overline{\alpha}) = UC_i^{\lambda_2}(A).$$

Thus, i^* is the agent whose claim is closest to $\overline{\alpha}$ from below. ■

It is well known that the single-crossing property of preferences is a sufficient condition for the existence of a majority voting equilibrium (e.g., Gans and Smart, 1996). Thus, we have the following corollary from Proposition 1.

Corollary 1 *There is a majority voting equilibrium for the* UC*-family of rules.*

Corollary 1 states that if we let teams vote for a rule within the UC-family, then there will be a majority winner. The identity of this winner will be problem specific and, thus, it will depend on the characteristics of the problem at stake. For problems with a distribution of claims skewed to the left, only the *uniform* rule is a majority winner. For problems with a distribution of claims skewed to the right, only *concede-and-divide* is a majority winner. For the remainder of the problems, each UC rule is a majority winner. This is a consequence of the fact that, as it can be inferred from (1), UC_i^{λ} is increasing (decreasing) in λ for agents with claims above (below) the average claim.

Another well-known consequence of the single-crossing property is that it guarantees progressivity comparisons of schedules (e.g., Jakobsson, 1976; Hemming and Keen, 1983). Thus, we can also obtain an interesting corollary from Proposition 1 in our setting, referring to the distributive power of the rules within the UC-family.

Formally, given $x, y \in \mathbb{R}^n$ satisfying $x_1 \leq x_2 \leq ... \leq x_n$, $y_1 \leq y_2 \leq ... \leq y_n$, and $\sum_{i=1}^n x_i = \sum_{i=1}^n y_i$, we say that x *is greater than* y *in the Lorenz ordering* if $\sum_{i=1}^k x_i \geq \sum_{i=1}^k y_i$, for each $k = 1, ..., n-1$, with at least one strict inequality. When x is greater than y in the Lorenz ordering, one can state (see, for instance, Dasgupta et al., 1973) that x is unambiguously "more egalitarian" than y. In our setting, we say that a rule R *Lorenz dominates* another rule R' if for each $A \in \mathcal{P}$, $R(A)$ is greater than $R'(A)$ in the Lorenz ordering. As the Lorenz criterion is a partial ordering, one might not expect to perform many comparisons of vectors. It turns out that, here, all rules within the family are fully ranked according to this criterion.

Corollary 2 *If* $0 \leq \lambda_1 \leq \lambda_2 \leq 1$ *then* UC^{λ_1} *Lorenz dominates* UC^{λ_2}*.*

Corollary 2 implies that the parameter defining the family can actually be interpreted as an index of the distributive power of the rules within the family.

4 Decomposing the benchmark family

In this section, we scrutinize the UC-family of rules further. We summarize first the performance of the rules within the family with respect to the other axioms introduced above[6].

Proposition 2 *A member of the* UC*-family of rules satisfies*

1. *non-negativity if and only if it is a member of the* UE*-family of rules*
2. *maximum aspirations if and only if it is a member of the* EC*-family of rules.*
3. *null team if and only if it is the equal-split rule.*
4. *essential team if and only if it is concede-and-divide.*

Combining Proposition 2 with Theorem 1, and noting that both *non-negativity* and *maximum aspirations* imply *weak upper bound*, additional characterizations are obtained as immediate corollaries.

Corollary 3 *The following statements hold:*

[6]The proof of Proposition 2, and some other results of the paper, can be found in the Appendix.

1. *A rule satisfies additivity, order preservation and non-negativity if and only if it is a member of the UE-family of rules.*

2. *A rule satisfies additivity, order preservation and maximum aspirations if and only if it is a member of the EC-family of rules.*

Corollary 3.2 is similar to Theorem 1 in Bergantiños and Moreno-Ternero (2020b), obtained by replacing *order preservation* with an axiom dubbed *symmetry*. The next result is a refinement of both as it shows that *equal treatment of equals*, which is weaker than *symmetry* and *order preservation*, closes the gap too.

Proposition 3 *A rule satisfies additivity, equal treatment of equals and maximum aspirations if and only if it is a member of the EC-family of rules.*

Finally, one infers from Proposition 2 that only the *equal-split* rule satisfies *non-negativity* and *maximum aspirations*. But one could also be interested in knowing the allocations satisfying both bounds for a given problem. This is what the next proposition states.

Proposition 4 *For each $A \in \mathcal{P}$, and each $i \in N$, $0 \leq UC_i^{\lambda}(A) \leq \alpha_i$ if and only if*

$$\alpha_i \geq \overline{\alpha} \quad \textit{and} \quad 0 \leq \lambda \leq \frac{n-2}{2(n-1)} + \frac{n\alpha_i}{2(n-1)(\alpha_i - \overline{\alpha})}.$$

$$\alpha_i \leq \overline{\alpha} \quad \textit{and} \quad \frac{n-2}{2(n-1)} - \frac{n\alpha_i}{2(n-1)(\overline{\alpha} - \alpha_i)} \leq \lambda \leq \frac{n-2}{2(n-1)(\overline{\alpha} - \alpha_i)}\overline{\alpha}.$$

In Table 1, we summarize the main results obtained for the benchmark family and its subfamilies.

Axioms\Rules	UC Th.1	UE Cor. 3	EC Cor. 3	EC Pr. 3	ES Pr. 2	CD Pr. 2
ETE				X		
OP	X	X	X		X	X
MA			X	X		
WUB	X				X	X
NN		X				
ADD	X	X	X	X	X	X
NT					X	
ET						X

Table 1: Characterizations of rules within the benchmark family

5 Beyond the benchmark family

In this section, we consider some combinations of axioms leading towards rules that extend the benchmark family studied in the previous sections. In Theorem 2, we characterize the rules satisfying *additivity*, *equal treatment of equals* and *weak upper bound.* In Theorem 3, we characterize the rules satisfying *additivity*, *equal treatment of equals* and *non-negativity.* In Theorem 4, we characterize the rules satisfying *additivity* and *order preservation.* In all cases, we obtain rules that are linear (but not necessarily convex) combinations of the *uniform rule* and *concede-and-divide*. For that reason, we conclude the section studying explicitly the performance of all the rules within the extended family $\{(1-\lambda)U + \lambda CD : \lambda \in (-\infty, +\infty)\}$ with respect to all the axioms, depending on λ.

The next result extends Theorem 1, weakening *order preservation* to *equal treatment of equals.*

Theorem 2 *A rule R satisfies additivity, equal treatment of equals and weak upper bound if and only if there exists $\lambda \in \left[1-\frac{n}{2}, 1\right]$ such that, for each $A \in \mathcal{P}$,*

$$R(A) = (1-\lambda)U(A) + \lambda CD(A).$$

Proof. As mentioned above, the *uniform rule* and *concede-and-divide* satisfy *additivity* and *equal treatment of equals*. It follows from there any linear combination of the two rules satisfies the two axioms too. As for *weak upper bound*, one can also show (after some algebraic computations) that, for each $\lambda \in \left[1 - \frac{n}{2}, 1\right]$, $(1-\lambda)U + \lambda CD$ satisfies it too.[7]

Conversely, let R be a rule satisfying the three axioms. Let $A \in \mathcal{P}$. For each pair $i, j \in N$, with $i \neq j$, let 1^{ij}, x and λ be defined as in the proof of Theorem 1. Using arguments similar to those used in the proof of Theorem 1 we can deduce that $R(A) = (1-\lambda)U(A) + \lambda CD(A)$.

By *weak upper bound*, $x \leq 1$ and $\frac{1-2x}{n-2} \leq 1$. Equivalently, $\frac{3-n}{2} \leq x \leq 1$. As $\lambda = \frac{nx-1}{n-1}$, it follows that $\lambda \in \left[1 - \frac{n}{2}, 1\right]$. ■

If instead of *weak upper bound* we consider *non-negativity*, we also have the following characterization result.

Theorem 3 *A rule R satisfies additivity, equal treatment of equals and non-negativity if and only if there exists $\lambda \in \left[\frac{-1}{n-1}, \frac{n-2}{2(n-1)}\right]$ such that, for each $A \in \mathcal{P}$,*

$$R(A) = (1-\lambda)U(A) + \lambda CD(A).$$

Proof. As mentioned above, any linear combination of the *uniform rule* and *concede-and-divide* satisfies *additivity* and *equal treatment of equals*. As for *non-negativity*, one can also show (after some algebraic computations) that, for each $\lambda \in \left[\frac{-1}{n-1}, \frac{n-2}{2(n-1)}\right]$, $(1-\lambda)U + \lambda CD$ satisfies it too.[8]

Conversely, let R be a rule satisfying the three axioms. Let $A \in \mathcal{P}$. For each pair $i, j \in N$, with $i \neq j$, let 1^{ij}, x and λ be defined as in the proof of Theorem 1. Using arguments similar to those used in the proof of Theorem 1 we can deduce that $R(A) = (1-\lambda)U(A) + \lambda CD(A)$.

By *non-negativity*, $x \geq 0$ and $\frac{1-2x}{n-2} \geq 0$. Equivalently, $0 \leq x \leq \frac{1}{2}$. As $\lambda = \frac{nx-1}{n-1}$, it follows that $\lambda \in \left[\frac{-1}{n-1}, \frac{n-2}{2(n-1)}\right]$. ■

We now explore the implications of the combination of *additivity* and *order preservation*.

Theorem 4 *A rule satisfies additivity and order preservation if and only there exists $\lambda \in [0, \infty)$ such that for each $A \in \mathcal{P}$,*

$$R(A) = (1-\lambda)U(A) + \lambda CD(A).$$

[7] See Remark 2 in the Appendix.

[8] See Remark 3 in the Appendix.

Proof. As mentioned above, any linear combination of the *uniform rule* and *concede-and-divide* satisfies *additivity*. As *concede-and-divide* satisfies *order preservation*, and the *uniform rule* assigns the same amount to all teams, it follows that $(1-\lambda)U + \lambda CD$ also satisfies *order preservation* for each $\lambda \in [0, \infty)$.

Conversely, let R be a rule satisfying the two axioms. Let $A \in \mathcal{P}$. For each pair $i, j \in N$, with $i \neq j$, let 1^{ij}, x and λ be defined as in the proof of Theorem 1. Using arguments similar to those used in the proof of Theorem 1 we can deduce that $R(A) = (1-\lambda)U(A) + \lambda CD(A)$.

By *order preservation*, $x \geq \frac{1-2x}{n-2}$ which implies that $x \geq \frac{1}{n}$. As $\lambda = \frac{nx-1}{n-1}$, it follows that $\lambda \in [0, +\infty)$. ■

We conclude this section with our more general result, which explores the implications of the combination of *additivity* and *equal treatment of equals*.

Theorem 5 *A rule satisfies additivity and equal treatment of equals if and only there exists* $\lambda \in (-\infty, +\infty)$ *such that, for each* $A \in \mathcal{P}$*,*

$$R(A) = (1-\lambda)U(A) + \lambda CD(A).$$

Proof. As mentioned above, any linear combination of the *uniform rule* and *concede-and-divide* satisfies *additivity* and *equal treatment of equals*. Conversely, let R be a rule satisfying the two axioms. Let $A \in \mathcal{P}$. For each pair $i, j \in N$, with $i \neq j$, let 1^{ij}, x and λ be defined as in the proof of Theorem 1. Using arguments similar to those used in the proof of Theorem 1 we can deduce that $R(A) = (1-\lambda)U(A) + \lambda CD(A)$. As no further axioms are considered, no bounds on the domain of λ can be imposed, from where it follows that $\lambda \in (-\infty, +\infty)$. ■

In Table 2, we summarize the main results obtained for the benchmark family and the extended families considered in this section. When we write $[a, b]$ at the top of a column, we refer to the family of rules $\{(1-\lambda)U + \lambda CD : \lambda \in [a, b]\}$.

Axioms\Rules	$[0,1]$ Th.1	$\left[1-\frac{n}{2},1\right]$ Th. 2	$\left[\frac{-1}{n-1},\frac{n-2}{2(n-1)}\right]$ Th. 3	$[0,+\infty)$ Th. 4	$(-\infty,+\infty)$ Th. 5
ETE		X	X		X
OP	X			X	
WUB	X	X			
NN			X		
ADD	X	X	X	X	X

Table 2: Beyond the benchmark family

We conclude this section studying the performance of all rules within the general family

$$\{(1-\lambda)U+\lambda CD : \lambda \in (-\infty,+\infty)\}$$

with respect to the axioms considered in this paper.

Proposition 5 *The following statements hold:*

(*a*) $(1-\lambda)U+\lambda CD$ *satisfies additivity for each* $\lambda \in (-\infty,+\infty)$.

(*b*) $(1-\lambda)U+\lambda CD$ *satisfies equal treatment of equals for each* $\lambda \in (-\infty,+\infty)$.

(*c*) $(1-\lambda)U+\lambda CD$ *satisfies order preservation if and only if* $\lambda \in [0,+\infty)$.

(*d*) $(1-\lambda)U+\lambda CD$ *satisfies weak upper bound if and only if* $\lambda \in \left[1-\frac{n}{2},1\right]$.

(*e*) $(1-\lambda)U+\lambda CD$ *satisfies maximum aspirations if and only if* $\lambda \in \left[\frac{n-2}{2(n-1)},1\right]$.

(*f*) $(1-\lambda)U+\lambda CD$ *satisfies non-negativity if and only if* $\lambda \in \left[\frac{-1}{n-1},\frac{n-2}{2(n-1)}\right]$.

(*g*) $(1-\lambda)U+\lambda CD$ *satisfies null team if and only if* $\lambda = \frac{n-2}{2(n-1)}$.

(*h*) $(1-\lambda)U+\lambda CD$ *satisfies essential team if and only if* $\lambda = 1$.

Proposition 5 can be summarized in the following table.

Axioms	$\lambda \in$
Additivity	$(-\infty, +\infty)$
Equal treatment of equals	$(-\infty, +\infty)$
Order preservation	$[0, +\infty)$
Weak upper bound	$\left[1 - \frac{n}{2}, 1\right]$
Maximum aspirations	$\left[\frac{n-2}{2(n-1)}, 1\right]$
Non-negativity	$\left[-\frac{1}{n-1}, \frac{n-2}{2(n-1)}\right]$
Null team	$\frac{n-2}{2(n-1)}$
Essential team	1

Table 3: Performance of the rules with respect to the axioms

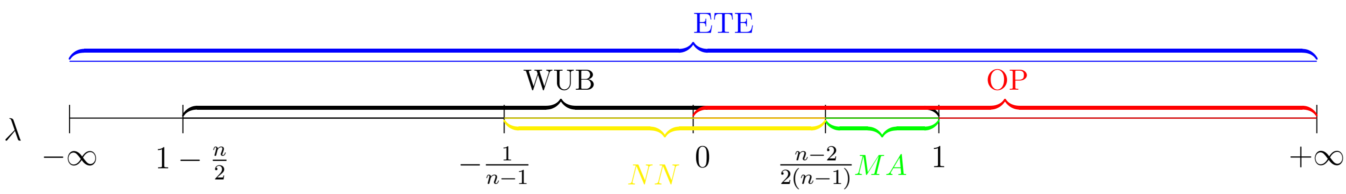

Figure 2. Performance of the rules with respect to the axioms.

6 Discussion

We have explored in this paper the axiomatic approach to the problem of sharing the revenues raised from the collective sale of broadcasting rights in sports leagues. We have uncovered the structure of this problem, setting the ground with three basic axioms: *additivity*, *order preservation* and *weak upper bound*. The combination of these axioms characterizes a large family of rules, which is made of compromises between the *uniform* rule and *concede-and-divide*, having the *equal-split* rule as a focal member. Thus, the family encompasses the three basic rules highlighted so far in this model. The family can be perfectly split in two, with the *equal-split* rule setting the limits for both parts, strengthening the *weak upper bound* in two opposite directions (*maximum aspirations* or *non-negativity*). If instead of strengthening the original axioms once weakens them in natural ways, extensions of the original family are characterized too.

Common to all of our characterization results is the axiom of *additivity*. This is an invariance requirement with a long tradition in axiomatic work (e.g., Shapley, 1953) but also considered strong under some circumstances. For results without *additivity* in this model, the reader is referred to Bergantiños and Moreno-Ternero (2020c).

All the families of rules we obtained are extremely well structured. They are all parametrized by a single element, which serves as an index of the distributive power of the rules. More precisely, once can fully rank in terms of the Lorenz dominance criterion the outcomes obtained by all the rules within each family, according to the parameter defining the family. Also, one can guarantee the existence of majority voting equilibria, when all teams are allowed to vote for the rule to share the broadcasting revenues, within each of these families. These two features are shared by some other one-parameter families of rules existing in the literature for related models (e.g., Moreno-Ternero and Villar, 2006a,b; Thomson, 2008; Moreno-Ternero, 2011; Thomson, 2019).

One could also be interested into approaching our problems with a (cooperative) game-theoretical approach, a standard approach in many related models of resource allocation (e.g., Littlechild and Owen, 1973; van den Nouweland et al., 1996; Ginsburgh and Zang, 2003). In Bergantiños and Moreno-Ternero (2020a), we associate to our problems a natural *optimistic* cooperative TU game in which, for each subset of teams, we define its worth as the total audience of the games played by the teams in that subset. The Shapley value (e.g., Shapley, 1953) of such a game yields the same solutions as the *equal-split* rule for the original problem. The egalitarian value (e.g., van den Brink, 2007) of that game yields the same solutions as the *uniform* rule. Casajus and Huettner (2013), van den Brink et al., (2013) and Casajus and Yokote (2019) characterize the family of values arising from the convex combination of the Shapley value and the egalitarian value. In our setting, this would correspond to the family of rules $\{UE^{\beta}\}_{\beta\in[0,1]}$ considered here. Thus, Corollary 3.1 in our paper could be considered as a parallel result to some of the results in that literature. No known value for TU-games is associated to concede-and-divide and, thus, no parallel characterization of the family of rules $\{UC^{\lambda}\}_{\lambda\in[0,1]}$ considered here can be obtained in the literature on TU-games.

References

[1] Bergantiños, G., Moreno-Ternero, J.D., 2020a. Sharing the revenues from broadcasting sport events. Management Science 66 (6), 2417-2431.

[2] Bergantiños, G., Moreno-Ternero, J.D., 2020b. Compromising to share the revenues from broadcasting sports leagues. R&R (twice) at Journal of Economic Behavior and Organization.

[3] Bergantiños, G., Moreno-Ternero, J.D., 2020c. Allocating extra revenues from broadcasting sports leagues. Journal of Mathematical Economics 90, 65-73.

[4] Brink van den, R., 2007. Null or nullifying players: The difference between the Shapley value and equal division solutions. Journal of Economic Theory 136, 767-775.

[5] Brink van den, R., Funaki Y., Ju, Y., 2013. Reconciling marginalism with egalitarianism: consistency, monotonicity, and implementation of egalitarian Shapley values, Social Choice and Welfare 40, 693-714.

[6] Casajus A., Huettner F., 2013. Null players, solidarity, and the egalitarian Shapley values. Journal of Mathematical Economics 49, 58-61.

[7] Casajus A., Yokote, K., 2019. Weakly differentially monotonic solutions for cooperative games. International Journal of Game Theory 48, 979-997.

[8] Dasgupta, P.S., Sen, A.K., Starret, D., 1973. Notes on the Measurement of Inequality. Journal of Economic Theory 6, 180-187.

[9] Gans, J. S., Smart, M., 1996. Majority voting with single-crossing preferences, Journal of Public Economics 59, 219-237.

[10] Ginsburgh, V., Zang, I., 2003. The museum pass game and its value. Games and Economic Behavior 43, 322-325.

[11] Hemming, R., Keen, M.J., 1983. Single-crossing conditions in comparisons of tax progressivity, Journal of Public Economics 20, 373-380.

[12] Jakobsson, U., 1976. On the measurement of the degree of progression, Journal of Public Economics 5, 161-168.

[13] Littlechild, S., Owen. G., 1973, A simple expression for the Shapley value in a special case. Management Science 20, 370-372.

[14] Moreno-Ternero J.D., 2011. Voting over piece-wise linear tax methods. Journal of Mathematical Economics 47, 29-36.

[15] Moreno-Ternero J.D., Villar A., 2006a. The TAL-family of rules for bankruptcy problems. Social Choice and Welfare 27, 231-249.

[16] Moreno-Ternero J.D., Villar A., 2006b. On the relative equitability of a family of taxation rules. Journal of Public Economic Theory 8, 283-291.

[17] van den Nouweland, A., Borm, P., van Golstein Brouwers, W., Groot Bruinderink, R., Tijs, S., 1996. A Game Theoretic Approach to Problems in Telecommunication. Management Science 42, 294-303.

[18] Shapley, L., 1953. A value for n-person games, in Contributions to the Theory of Games II (Annals of Mathematics Studies 28), ed. by H.W. Kuhn and A.W. Tucker, Princeton: Princeton University Press, 307-317.

[19] Thomson, W., 2008. Two families of rules for the adjudication of conflicting claims. Social Choice and Welfare 31, 667-692.

[20] Thomson W., 2019. How to divide when there isn't enough: from Aristotle, the Talmud, and Maimonides to the axiomatics of resource allocation, Econometric Society Monograph. Cambridge University Press.

To save space, we have included in this appendix, which is not necessarily intended for publication, some technical aspects of our analysis, as well as secondary proofs.

7 Appendix

7.1 Missing proofs

Remark 1 *The axioms of Theorem 1 are independent.*

For each $A \in \mathcal{P}$, and each $i \in N$, we define the rule R^1 as

$$R_i^1(A) = \begin{cases} U(A) & \text{if } ||A|| \leq 100 \\ CD(A) & \text{if } ||A|| > 100. \end{cases}$$

R^1 satisfies all axioms in the theorem but additivity.

Let R^2 be defined as follows. For each $\{i,j\} \in N$ and $k \in N$ we define

$$R_k^2\left(1^{ij}\right) = \begin{cases} 0 & \text{if } k \in \{i,j\} \\ \frac{1}{n-2} & \text{otherwise} \end{cases}$$

We extend R^2 to each problem A using additivity. Namely, $R^2(A) = \sum_{\{i,j\} \subset N} a_{ij} R^2\left(1^{ij}\right)$. R^2 satisfies all axioms in the theorem but order preservation.

Let R^3 be defined as follows. For each $\{i,j\} \in N$ and $k \in N$ we define

$$R_k^3\left(1^{ij}\right) = \begin{cases} 2 & \text{if } k \in \{i,j\} \\ \frac{-3}{n-2} & \text{otherwise} \end{cases}$$

We extend R^3 to each problem A using additivity. Namely, $R^3(A) = \sum_{\{i,j\} \subset N} a_{ij} R^3\left(1^{ij}\right)$. R^3 satisfies all axioms in the theorem but weak upper bound.

Proof of Proposition 2. For statement 1, it is obvious that all rules within the UE-family satisfy *non-negativity* because both U and ES do so. We have seen above that the UC-family is the union of the UE-family and the EC-family. Besides, ES is the only rule in the UC-family belonging to the UE-family and the EC-family. Thus, we only need to show that any rule within the EC-family, except for ES, violates that axiom. To do so, consider, for instance, the problem in which

$$A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 150 \\ 0 & 150 & 0 \end{pmatrix}.$$

As $CD_1(A) = -300$ and $ES_1(A) = 0$, it follows that $(1-\gamma)ES_1(A) + \gamma CD_1(A) < 0$, for each $\gamma \in (0, 1]$.

For statement 2, Bergantiños and Moreno-Ternero (2020b) prove that any rule in the EC-family satisfies *maximum aspirations*. Thus, it is enough to show that any rule within the UE-family, except for ES, violates that axiom. The previous example would be valid for that too. Note that $U_1(A) = 100$ and $\alpha_1 = 0$. As $ES_1(A) = 0$, it follows that $(1-\beta)U_1(A) + \beta ES_1(A) > \alpha_1$, for each $\beta \in [0, 1)$.

As for statements 3 and 4, they are a straightforward consequence of Theorem 1 in Bergantiños and Moreno-Ternero (2020a), and the fact that *order preservation* implies *equal treatment of equals*. □

Proof of Proposition 3. It is similar to the proof of Theorem 1 in Bergantiños and Moreno-Ternero (2020b). □

Proof of Proposition 4.

Let $A \in \mathcal{P}$, $i \in N$, and $\lambda \in [0, 1]$. Then, $UC_i^\lambda(A) \geq 0$ if and only if

$$\frac{||A||}{n} + \lambda \frac{n-1}{n-2} (\alpha_i - \overline{\alpha}) \geq 0.$$

Equivalently, as $||A|| = \frac{n\overline{\alpha}}{2}$,

$$(n-2)\,\overline{\alpha} + 2\lambda\,(n-1)\,(\alpha_i - \overline{\alpha}) \geq 0,$$

i.e.,

$$\lambda\,(\alpha_i - \overline{\alpha}) \geq -\frac{n-2}{2\,(n-1)}\overline{\alpha}.$$

Now, if $\alpha_i \geq \overline{\alpha}$, the above holds trivially. If, instead, $\alpha_i < \overline{\alpha}$, then the above is equivalent to

$$\lambda \leq \frac{n-2}{2\,(n-1)\,(\overline{\alpha} - \alpha_i)}\overline{\alpha},$$

as desired.

Now, $UC_i^\lambda(A) \leq \alpha_i$ if and only if

$$\frac{||A||}{n} + \lambda \frac{n-1}{n-2} (\alpha_i - \overline{\alpha}) \leq \alpha_i.$$

Equivalently,

$$\lambda\,(\alpha_i - \overline{\alpha}) \leq \frac{(n-2)\,(\alpha_i - \overline{\alpha}) + n\alpha_i}{2\,(n-1)}.$$

If $\alpha_i \geq \overline{\alpha}$, the above is equivalent to

$$\lambda \leq \frac{n-2}{2(n-1)} + \frac{n\alpha_i}{2(n-1)(\alpha_i - \overline{\alpha})}.$$

If $\alpha_i < \overline{\alpha}$, the above is equivalent to

$$\lambda \geq \frac{n-2}{2(n-1)} - \frac{n\alpha_i}{2(n-1)(\overline{\alpha} - \alpha_i)}.$$

□

Remark 2 *$(1-\lambda)U + \lambda CD$ satisfies weak upper bound, for each $\lambda \in \left[1 - \frac{n}{2}, 1\right]$.*

Let $A \in \mathcal{P}$, $i \in N$, and $\lambda \in \left[1 - \frac{n}{2}, 1\right]$. Then, $(1-\lambda)U_i(A) + \lambda CD_i(A) \leq ||A||$ if and only if

$$\frac{\overline{\alpha}}{2} + \lambda \frac{n-1}{n-2}(\alpha_i - \overline{\alpha}) \leq \frac{n\overline{\alpha}}{2}.$$

Equivalently,

$$\lambda(\alpha_i - \overline{\alpha}) \leq \frac{n-2}{2}\overline{\alpha}. \tag{3}$$

We consider three cases.

1. *$\alpha_i = \overline{\alpha}$. Then (3) obviously holds.*

2. *$\alpha_i > \overline{\alpha}$. Then (3) is equivalent to*

$$\lambda \leq \frac{(n-2)\overline{\alpha}}{2(\alpha_i - \overline{\alpha})}.$$

As $\lambda \leq 1$ it is enough to prove that

$$1 \leq \frac{(n-2)\overline{\alpha}}{2(\alpha_i - \overline{\alpha})}.$$

Equivalently,

$$2(\alpha_i - \overline{\alpha}) \leq (n-2)\overline{\alpha}$$

which holds because $\alpha_i \leq ||A|| = \frac{n\overline{\alpha}}{2}$.

3. *$\alpha_i < \overline{\alpha}$. Then (3) is equivalent to*

$$\lambda \geq \frac{(n-2)\overline{\alpha}}{2(\alpha_i - \overline{\alpha})}.$$

As $\lambda \geq 1 - \frac{n}{2}$, it is enough to prove that

$$1 - \frac{n}{2} \geq \frac{(n-2)\overline{\alpha}}{2(\alpha_i - \overline{\alpha})}.$$

Equivalently,

$$\overline{\alpha} - \alpha_i \leq \overline{\alpha},$$

which obviously holds.

Remark 3 $(1-\lambda)U + \lambda CD$ *satisfies non-negativity, for each* $\lambda \in \left[\frac{-1}{n-1}, \frac{n-2}{2(n-1)}\right]$.

Let $A \in \mathcal{P}$, $i \in N$, *and* $\lambda \in \left[\frac{-1}{n-1}, \frac{n-2}{2(n-1)}\right]$. *Then,* $(1-\lambda)U_i(A) + \lambda CD_i(A) \geq 0$ *if and only if*

$$\frac{\overline{\alpha}}{2} + \lambda \frac{n-1}{n-2} \left(\alpha_i - \overline{\alpha}\right) \geq 0.$$

Equivalently,

$$\lambda \left(\alpha_i - \overline{\alpha}\right) \geq \frac{-\left(n-2\right)\overline{\alpha}}{2(n-1)} \tag{4}$$

We consider three cases.

1. $\alpha_i = \overline{\alpha}$. *Then* (4) *obviously holds.*

2. $\alpha_i > \overline{\alpha}$. *Then* (4) *is equivalent to*

$$\lambda \geq \frac{-\left(n-2\right)\overline{\alpha}}{2(n-1)(\alpha_i - \overline{\alpha})}.$$

As $\lambda \geq \frac{-1}{n-1}$, *it is enough to prove that*

$$\frac{1}{n-1} \leq \frac{\left(n-2\right)\overline{\alpha}}{2(n-1)(\alpha_i - \overline{\alpha})}.$$

Equivalently,

$$2\alpha_i \leq n\overline{\alpha},$$

which holds because $\alpha_i \leq ||A|| = \frac{n\overline{\alpha}}{2}$.

3. $\alpha_i < \overline{\alpha}$. *Then* (4) *is equivalent to*

$$\lambda \leq \frac{-\left(n-2\right)\overline{\alpha}}{2(n-1)(\alpha_i - \overline{\alpha})}.$$

As $\lambda \leq \frac{n-2}{2(n-1)}$, *it is enough to prove that*

$$\frac{n-2}{2(n-1)} \leq \frac{\left(n-2\right)\overline{\alpha}}{2(n-1)(\overline{\alpha} - \alpha_i)}.$$

Equivalently,

$$\overline{\alpha} - \alpha_i \leq \overline{\alpha},$$

which obviously holds.

Proof of Proposition 5.

Statements (a) and (b) are trivial.

As for statement (c), and as mentioned in the proof of Theorem 4, as *concede-and-divide* satisfies *order preservation*, and the *uniform rule* assigns the same amount to all teams, it follows that $(1-\lambda)U+\lambda CD$ also satisfies *order preservation* for each $\lambda \in [0,+\infty)$. It remains to show that the property is violated for any $\lambda \in (-\infty, 0)$. Consider the same problem as above in which

$$A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 150 \\ 0 & 150 & 0 \end{pmatrix}.$$

As $U(A) = (100, 100, 100)$ and $CD(A) = (-300, 300, 300)$, it follows that $(1-\lambda)U_1(A) + \lambda CD_1(A) = 100 - 400\lambda > 100 + 200\lambda = (1-\lambda)U_2(A) + \lambda CD_2(A)$, for each $\lambda \in (-\infty, 0)$.

As for statement (d), and as shown in Remark 2, $(1-\lambda)U+\lambda CD$ satisfies *weak upper bound* for each $\lambda \in \left[1-\frac{n}{2}, 1\right]$. It remains to show that the property is violated for any $\lambda \notin \left[1-\frac{n}{2}, 1\right]$. To do so, aonsider, again, the same problem as above. Then, as $U(A) = (100, 100, 100)$ and $CD(A) = (-300, 300, 300)$, it follows that $(1-\lambda)U_2(A)+\lambda CD_2(A) = 100+200\lambda > 300 = ||A||$, for each $\lambda > 1$. Similarly, $(1-\lambda)U_1(A) + \lambda CD_1(A) = 100 - 400\lambda > 300 = ||A||$, for each $\lambda < -\frac{1}{2} = 1 - \frac{n}{2}$.

As for statement (e), and as shown in Proposition 2, $(1-\lambda)U+\lambda CD$ satisfies *maximum aspirations* for each $\lambda \in \left[\frac{n-2}{2(n-1)}, 1\right]$.[9] It is also shown therein that for each $\lambda \in \left[0, \frac{n-2}{2(n-1)}\right)$, the property is violated. Thus, it remains to show that the property is also violated for any $\lambda \notin [0, 1]$. To do so, consider, again, the same problem as above. Then, as $U(A) = (100, 100, 100)$ and $CD(A) = (-300, 300, 300)$, it follows that $(1-\lambda)U_1(A) + \lambda CD_1(A) = 100 - 400\lambda > 0 = \alpha_1$, for each $\lambda < 0$. Similarly, $(1-\lambda)U_2(A) + \lambda CD_2(A) = 100 + 200\lambda > 300 = \alpha_2$, for each $\lambda > 1$.

As for statement (f), and as shown in Remark 3, $(1-\lambda)U+\lambda CD$ satisfies *non-negativity* for each $\lambda \in \left[\frac{-1}{n-1}, \frac{n-2}{2(n-1)}\right]$. As shown in Proposition 2, for each $\lambda \in \left(\frac{n-2}{2(n-1)}, 1\right]$, the property is violated.[10] Thus, it remains to show that the property is also violated for any $\lambda \notin \left[\frac{-1}{n-1}, 1\right]$. To do so, consider, again, the same problem as above. Then, as $U(A) = (100, 100, 100)$ and $CD(A) = (-300, 300, 300)$, it follows that $(1-\lambda)U_1(A) + \lambda CD_1(A) = 100 - 400\lambda < 0$, for each $\lambda > 1$. Similarly, $(1-\lambda)U_2(A) + \lambda CD_2(A) = 100 + 200\lambda < 0$, for each $\lambda < -\frac{1}{2} = -\frac{1}{n-1}$.

[9] Note that those rules correspond precisely with the UE-family of rules.

[10] Note that those rules correspond precisely with the EC-family of rules.

Statements (g) and (h) are straightforward consequences of Proposition 2. $\square$

7.2 Extra material

We now study which specific rule within the UC-family could be a majority winner for each problem. We obtain three different scenarios, depending on the characteristics of the problem at stake. For some problems, only the *uniform* rule is a majority winner. For some other problems, only *concede-and-divide* is a majority winner. For the remainder of the problems, each rule within the family is a majority winner.

For each $A \in \mathcal{P}$, we consider the following partition of N, with respect to the average claim ($\bar{\alpha}$): $N_l(A) = \{i \in N : \alpha_i < \bar{\alpha}\}$, $N_u(A) = \{i \in N : \alpha_i > \bar{\alpha}\}$, and $N_e(A) = \{i \in N : \alpha_i = \bar{\alpha}\}$. That is, taking the average claim (within the tournament) as the benchmark threshold, we consider three groups referring to individuals with claims below, above, or exactly at, the threshold. When no confusion arises, we simply write N_l, N_u, and N_e. Note that $n = |N_l| + |N_u| + |N_e|$.

Proposition 6 *Let $A \in \mathcal{P}$. The following statements hold:*

(i) *If $2|N_l| > n$, then $U(A)$ is the unique majority winner.*

(ii) *If $2|N_u| > n$, then $CD(A)$ is the unique majority winner.*

(iii) *Otherwise, each $UC^{\lambda}(A)$ is a majority winner.*

Proof. Let $0 \leq \lambda \leq 1$, and $A \in \mathcal{P}$. For each $i \in N$,

$$UC_i^{\lambda}(A) = \frac{||A||}{n} + \lambda \frac{n-1}{n-2}(\alpha_i - \overline{\alpha}).$$

If $\alpha_i > \bar{\alpha}$, then $UC_i^{\lambda}(A)$ is an increasing function of λ, thus maximized at $\lambda = 1$. This implies that, for each $i \in N_u$, $CD_i(A)$ is the most preferred outcome (among those provided by the family).

If $\alpha_i < \bar{\alpha}$, then $UC_i^{\lambda}(A)$ is a decreasing function of λ, thus maximized at $\lambda = 0$. This implies that, for each $i \in N_l$, $U_i(A)$ is the most preferred outcome (among those provided by the family).

If $\alpha_i = \bar{\alpha}$, then $UC_i^{\lambda}(A) = \frac{||A||}{n}$ for each $\lambda \in [0,1]$. This implies that, for each $i \in N_e$, all rules in the family yield the same outcome.

From the above, statements (i) and (ii) follow trivially. Assume, by contradiction, that statement (iii) does not hold. Then, there exists $A \in \mathcal{P}$ and $\lambda \in [0,1]$ such that UC^{λ} is not a majority winner for A. Thus, we can find $\lambda' \in [0,1]$ such that $UC_i^{\lambda'}(A) > UC_i^{\lambda}(A)$ holds for the majority of the teams. We then consider two cases:

Case $\lambda' > \lambda$.

In this case, $UC_i^{\lambda'}(A) > UC_i^{\lambda}(A)$ if and only if $i \in N_l$. Now,

$$\begin{aligned} |N_l| &= \left|\left\{i \in N : UC_i^{\lambda'}(A) > UC_i^{\lambda}(A)\right\}\right| \\ &> \left|\left\{i \in N : UC_i^{\lambda'}(A) \leq UC_i^{\lambda}(A)\right\}\right| \\ &= |N_u| + |N_e| \end{aligned}$$

which is a contradiction.

Case $\lambda' < \lambda$.

In this case, $UC_i^{\lambda'}(A) > UC_i^{\lambda}(A)$ if and only if $i \in N_u$. Now,

$$\begin{aligned} |N_u| &= \left|\left\{i \in N : UC_i^{\lambda'}(A) > UC_i^{\lambda}(A)\right\}\right| \\ &> \left|\left\{i \in N : UC_i^{\lambda'}(A) \leq UC_i^{\lambda}(A)\right\}\right| \\ &= |N_l| + |N_e| \end{aligned}$$

which is a contradiction. ■

Proposition 6 implies that if the distribution of claims is skewed to the left (i.e., the median claim is below the mean claim), then the *uniform* allocation (the most equal allocation within the family) is the majority winner, whereas if it is skewed to the right (i.e., the median claim is above the mean claim), then the *concede-and-divide* allocation (the most unequal allocation within the family, as proved below) is the majority winner. If it is not skewed, then any *compromise* allocation can be a majority winner.

The single-crossing property also guarantees that the social preference relationship obtained under majority voting is transitive, and corresponds to the median voter's. In our setting, the median voter corresponds to the team with the median overall audience (claim). Thus, depending on whether this median overall audience is below or above the average audience, the median voter's preferred rule (and, thus, the majority winner) will either be the *uniform* rule or *concede-and-divide*. In other words, a tournament with a small number of very strong teams (i.e., with very high claims in relative terms) will proclaim the *uniform* allocation (the one favoring weaker teams more within the family) as the majority winner, whereas a tournament with a small number of very weak teams (i.e., with very small claims in relative terms) will proclaim the *concede-and-divide* allocation (the one favoring stronger teams more within the family).

Corollary 4 *Let $A \in \mathcal{P}$ be such that n is odd. The following statements hold:*

(i) If $\alpha_m < \bar{\alpha}$, then $U(A)$ is the unique majority winner.

(ii) If $\alpha_m > \bar{\alpha}$, then $CD(A)$ is the unique majority winner.

(iii) If $\alpha_m = \bar{\alpha}$, then any $UC^{\lambda}(A)$ is a majority winner.

Proof. If $\alpha_m < \bar{\alpha}$, then $|N_l| \geq m$. Hence $2|N_l| > n$. By Proposition 6, statement (i) holds.

If $\alpha_m > \bar{\alpha}$, then $|N_u| \geq m$. Hence $2|N_u| > n$ By Proposition 6, statement (ii) holds.

If $\alpha_m = \bar{\alpha}$, then $|N_l| < m$, $|N_u| < m$, and $|N_e| > 0$. Hence, we are in case (iii) of the statement of Proposition 6, which concludes the proof. ■

Corollary 5 *Let $A \in \mathcal{P}$ be such that n is even. The following statements hold:*

(i) If $\alpha_{\frac{n+2}{2}} < \bar{\alpha}$, then $U(A)$ is the unique majority winner.

(ii) If $\alpha_{\frac{n}{2}} > \bar{\alpha}$, then $CD(A)$ is the unique majority winner.

(iii) If $\alpha_{\frac{n}{2}} \leq \bar{\alpha} \leq \alpha_{\frac{n+2}{2}}$, then any $UC^{\lambda}(A)$ is a majority winner.

Proof. If $\alpha_{\frac{n+2}{2}} < \bar{\alpha}$, then $|N_l| \geq \frac{n+2}{2}$. Hence $2|N_l| > n$. By Proposition 6, statement (i) holds.

If $\alpha_{\frac{n}{2}} > \bar{\alpha}$, then $|N_u| \geq \frac{n+2}{2}$. Hence $2|N_u| > n$. By Proposition 6, statement (ii) holds.

Suppose now that $\alpha_{\frac{n}{2}} \leq \bar{\alpha} \leq \alpha_{\frac{n+2}{2}}$. Then, it is enough to prove that we are in case (iii) of the statement of Proposition 6. That is, we have to prove that neither $|N_l| > |N_u| + |N_e|$ nor $|N_u| > |N_l| + |N_e|$ hold. We consider several subcases:

1. If $\bar{\alpha} = \alpha_{\frac{n}{2}}$, then $|N_l| < \frac{n}{2}$, $|N_u| \leq \frac{n}{2}$ and $|N_e| > 0$.

2. If $\alpha_{\frac{n}{2}} < \bar{\alpha} < \alpha_{\frac{n+2}{2}}$, then $|N_l| = \frac{n}{2}$, $|N_u| = \frac{n}{2}$ and $|N_e| = 0$.

3. If $\bar{\alpha} = \alpha_{\frac{n+2}{2}}$, then $|N_l| \leq \frac{n}{2}$, $|N_u| < \frac{n}{2}$ and $|N_e| > 0$.

In either case, the desired conclusion holds. ■