CEIS TOR VERGATA Research Papers

Università di Roma Tor Vergata

ISSN 2610-931X

CEIS Tor Vergata

RESEARCH PAPER SERIES

Vol. 24, Issue 5, No. 632 – November 2026

Endogenous Sampling: Itô Chaos and Limit Theory

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Endogenous sampling: Itô chaos and limit theory*

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November 16, 2026

Abstract

Economic theory suggests that the observation arrival times should be endogenous to the fundamental price process. We introduce *chaotic representations* of the endogenous durations between arrival times and show that they facilitate asymptotic theory on realized measures, beyond realized variance. We work with forms of hitting-time sampling studied in the literature as well as with new hitting-time specifications defined with respect to *spot* variance, random clocks, and beyond. In all cases, the interaction between the derived *chaotic representations* of the durations and the assumed filtration (representing layers of process-specific information, like in a *chaotic representation*) leads to economically-transparent asymptotic arguments. Relying on *chaotic representations* and asymptotic arguments, we discuss notions of *weak* and *strong* endogeneity.

Keywords: High-frequency data, realized measures, endogenous sampling, Itô chaos

JEL Classification: C14, C58

*We thank the participants in the "Barcelona Workshop in Financial Econometrics" (Barcelona, May 22-23, 2025), the "Tunisia Conference on Financial Econometrics" (Hammamet, June 16-18, 2025), the "High-Voltage Econometrics V" Conference (Turin, October 2-3, 2025), the 18th Annual SoFie Conference (Macau, June 17-19, 2026), and the CIREQ Financial Econometrics Conference (Montreal, September 11-12, 2026). We are particularly grateful to Per Mykland for extensive discussions and to Marcel Rindisbacher (discussant) for his many suggestions.

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1 Introduction

It is an empirical fact that the durations between price updates change over time (Hautsch, 2011, provides a comprehensive review and discussion). Sound economic theory suggests that these changes are necessarily endogenous, in that they should depend on the dynamics of the underlying efficient price and/or its functionals (volatility, *inter alia*).[1] An immediate consequence of this observation is that the econometrics of high-frequency data should account for a sampling/observation process[2] which is both *random* and *endogenous*.

As an example of economic logic supporting both features, consider a typical economy in which informed agents trade only when the trade is profitable net of execution costs (c.f., e.g., Bandi et al., 2017). Fig. 1 represents an efficient price (i.e., a discounted expectation of future cash flows conditional on the informed agents' information set) fluctuating within bid and ask prices. The informed agents will trigger a buy (resp. sell) order only when the efficient price is higher (resp. lower) than the ask (resp. bid) price. In essence, information-based trading leads to both *random* durations between trade arrivals and *endogeneity* between the arrivals and the level of the underlying efficient price process.

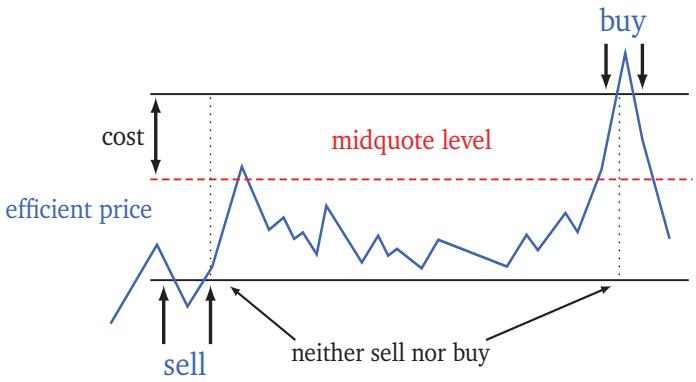

Figure 1: Information-based trading.

In spite of economic logic and empirical evidence, allowance for (forms of) endogenous sampling in the computation of realized measures using high-frequency data is - to this day - rare and far from being the norm. Fundamental contributions in this area are, e.g., Fukasawa (2010b), Hayashi et al. (2011), Fukasawa and Rosenbaum (2012), Robert and Rosenbaum (2012), and Li et al. (2013)[3]. As is natural, the focus of these early works is on realized variance, the traditional object of interest in high-frequency econometrics. Since then, i.e., in the last fifteen years, limited progress has been made on the subject, deterministic or random (but exogenous) sampling remaining the typical sampling schemes in the literature. Accounting for endogeneity is known to be complex. While the reason for ignoring an important feature of the data is - understandably - technical convenience, the cost is asymptotic results that may depart, in important ways, from realistic representations.

This is true, in particular, for realized measures - beyond realized variance - that have been at the center of the literature over its more recent history. It is, in fact, a fundamental tenet of stochastic process theory that - provided efficient prices are not latent - realized variance converges to quadratic variation as long as the supremum of the distance between observations vanishes (e.g., Jacod and Shiryaev, 2003, Theorem I.4.47). In other words, the presence of endogeneity in the observations arrival process does not affect the realized variance probability limit. It only affects its limiting law. In this sense, realized variance may be viewed as special. In general, alternative realized measures, e.g. higher-order moments, have limiting laws *as well as* probability limits which are, instead, influenced by the nature of the sampling scheme, even when efficient prices are observable.[4] As we show below, the impact of endogeneity may be substantial both on probability limits and on limiting laws.

[1] The negative dependence between volatility and duration is a robust stylized fact reported, e.g., in Engle (2000) and Renault and Werker (2011).

[2] As we discuss below, the reader may view this article as modeling the observation arrival times as well as alternative sampling rules, as in e.g. Li et al. (2025). We will make an explicit distinction between the two when needed.

[3] In a related piece, Li et al. (2014) extend the framework to include market microstructure noise in integrated volatility estimation. The bivariate case is studied, e.g., in Robert and Rosenbaum (2012) and Potiron and Mykland (2017).

[4] Beyond realized variance, Bandi et al. (2023) study the heterogeneous impact of endogeneity on general power variation measures in a model in which endogeneity is generated by the (possible) dependence between the likelihood of lack of price updates and the volatility of the unobserved efficient price process.

Against this backdrop, our objective in this article is to provide methods which would facilitate accounting for economically-reasonable forms of endogenous sampling while broadening the scope of the econometrics of endogenous sampling to modern realized measures, beyond realized variance. To this end, we make several contributions.

First, the joint moments of the price process and the endogenous durations are central to the derivation of limiting results for realized measures under endogeneity. Mykland's Bartlett-type identities for martingales (Mykland, 1994) provide a convenient way to handle these joint moments. We present an (almost) elementary way to define these identities (automatically, to any order) using fundamental properties of polynomial (Hermite) martingales. We discuss the mapping between Brownian Bartlett-type identities (i.e., identities for joint moments of the random durations and Brownian motion changes) and *chaotic representations* of powers of Brownian motion.

Second, we turn to the derivation of stable laws. When endogeneity is accounted for, alternative approaches to the derivation of stable limits co-exist. Hayashi et al. (2011) make use of the classical approach in Jacod (2008) for partial sums of triangular arrays and stable limits are proven on a suitable sub-filtration of the original filtration. As discussed in Hayashi et al. (2011), the adopted sub-flitration may exclude cases of endogeneity of economic interest (e.g., it excludes cases of hitting-time sampling discussed in this article). Li et al. (2013), on the other hand, employ a martingale approach to stable laws (c.f. Theorem 2.28 in Mykland and Zhang, 2012) and the filtration on which a stable central limit theorem is derived is one that is generated by a finite number of Brownian motion.[5] This latter approach allows for more general forms of endogeneity, but it may be viewed as being less transparent regarding the derivation of limiting biases and variances.

We reconcile the logical transparency of stable convergence results relying on triangular arrays (as in Jacod, 2008, and the bulk of the high-frequency literature after him) with the appeal of the information set used in the martingale approach of, e.g., Li et al. (2013).[6] Focusing on the *tricity*, as a maintained object of econometric interest throughout this article up to Section 4, we show how to apply Jacod's approach with respect to the Brownian filtration in Li et al. (2013), thereby combining methodological transparency with forms of endogeneity of economic interest.

The key to our approach is a *chaotic representation* of the endogenous durations between price updates. The representation is central to proving the five conditions in Jacod's stable central limit theorem (c.f. Podolskij and Vetter, 2010, for a revealing discussion of these conditions). The fifth condition (i.e., Eq. (10) in Theorem 1 of Podolskij and Vetter, 2010), in particular, is a long-run orthogonality property of the elements of the triangular array with respect to all bounded martingales (defined on the Brownian filtration) that are themselves orthogonal with respect to the driving Brownian motion(s). The condition justifies the representation of the limiting law in terms of driving Brownian motion(s) and an independent (of them) Brownian motion defined on a good extension of the original probability space. In problems in which sampling is random, this condition generally reduces to proving the long-run orthogonality between the random durations and the above-described bounded martingales. We show how the proposed *chaotic representation* of the durations assists in doing so.

We derive *chaotic representations* of the durations between price updates for typical, as well as novel, forms of endogenous sampling. We consider hitting-time sampling with respect to (1) the driving price Brownian motion, (2) the price process, (3) the *integrated* variance of the price process, and (4) the *spot* variance of the price process. We show how the interaction between the Brownian filtration (which may be interpreted as representing layers of process-specific information, like in a *chaotic representation*) and the *chaotic representation* of the durations provides an economically-transparent way to derive and interpret limiting arguments. Using this interaction, we introduce notions of *strong* and *weak* endogeneity. The former is defined as an impact on the stable law of the quantity of econometric interest (the *tricity*, in our case) with an additional impact on the probability limit relative to the non-endogenous case. The latter is solely defined as an impact on the stable law without an endogeneity-induced asymptotic bias. Because the *chaotic representation* of the durations formalizes their stochastic nature, it provides a justification for both forms of endogeneity in terms of depth of the information flow. Inspection of both the stochastic order and the layer of information of each element in the *chaotic representation* will - in fact - reveal and justify the nature of the endogeneity and its impact on the asymptotic centering and distribution of the estimator.

One may view our durations, and their *chaotic representations*, as primitive *structural* assumptions on the generation of observations (as in the example in Fig. 1) or, alternatively, as the consequence of economically-specified sampling rules as, e.g., in the recent work of Li et al. (2025). In both cases the representations may be

[5] As discussed in Mykland and Zhang (2012), the use - on suitable spaces - of such filtration in order to turn standard convergence in law to stable convergence in law is justified in Jacod and Protter (1998). Below, we discuss the economic significance of the filtration and its interpretation within chaos theory.

[6] See, also, Potiron and Mykland (2017).

used as a *low-level* assumption in the derivation of limiting results, thereby replacing *high-level* assumptions on the convergence of moments used, e.g., in Fukasawa (2010b) and Li et al. (2013).

In Section 4, we turn to the uncertainty zone model proposed by Robert and Rosenbaum (2012). By combining a *latent* efficient price process with *observed* prices on a grid, the model is designed to capture stylized properties of recorded high-frequency financial prices, in particular rounding, price discontinuities, bid-ask bounce effects, durations inversely correlated with the efficient price volatility as well as endogeneity between the price arrivals and the efficient price process. Again, we show how to express the corresponding durations in terms of their *chaotic representation* and use the representation to facilitate considerably the derivation of limiting arguments. We note that, in this section, we return to the traditional object of interest, namely realized variance. We do so for two reasons. First, efficient prices are latent. Therefore, Theorem I.4.47 in Jacod and Shiryaev, 2003 does not apply and realized variance has, in general, a probability limit which is affected by latency. Second, by focusing on realized variance, we may relate our results to existing results in the literature that, like in our case, have used rounded prices on a grid to define realized variance but, differently from us, have worked under deterministic (rather than random and endogenous) sampling, e.g., Jacod (1996), Delattre and Jacod (1997), and Li and Mykland (2015).

We proceed as follows. Section 2 reviews Mykland's Bartlett-type identities for martingales before providing an alternative derivation that will assist usage and interpretability. We discuss the conceptual map between the identities and *chaotic representations* of powers of Brownian motion. In Section 3 we turn to stable laws (for the *tricity*) in the presence of four alternative forms of endogenous hitting-time sampling. Using the stochastic order and layer of information of individual components of the *chaotic representations* of the durations we derive implications for weak and strong endogeneity. The section concludes with a simple taxonomy associating the nature of the sampling scheme to the type of endogeneity (c.f. Table 2). Section 4 derives a *chaotic representation* of the durations in the uncertainty zone model of Robert and Rosenbaum (2012). We use the representation to study the impact of price rounding on realized variance when sampling is endogenous and the interaction between durations and rounding itself cannot be ignored. Section 5 concludes. Proofs are contained in the Appendix.

2 Bartlett-type identities

Bartlett identities have a long history in likelihood-based estimation (Bartlett, 1953a,b). Bartlett-type identities have also been introduced for martingales (Mykland, 1994). After briefly reviewing the traditional approach in Bartlett (1953a,b) and the results in Mykland (1994), we present an (almost) elementary way to derive the latter using properties of polynomial (Hermite) martingales. We discuss how Mykland's Bartlett-type identities can facilitate asymptotic theory in high-frequency financial econometrics in the presence of endogenous sampling, an underexplored - but economically warranted - direction in the literature.

We begin with a succinct review of Bartlett's traditional results. Write the log-likelihood function as $\ell(\theta)$. (In the notation, for conciseness, we ignore explicit dependence on data.)

- The first identity:

$$\mathbb{E}\left[\frac{\partial \ell(\theta_0)}{\partial \theta}\right] = 0$$

- The second identity:

$$\mathbb{E}\left[\left(\frac{\partial \ell(\theta_0)}{\partial \theta}\right)^2\right] = -\mathbb{E}\left[\frac{\partial^2 \ell(\theta_0)}{\partial \theta^2}\right]$$

- The third identity:

$$\mathbb{E}\left[\left(\frac{\partial \ell(\theta_0)}{\partial \theta}\right)^3\right] = -\mathbb{E}\left[\frac{\partial^3 \ell(\theta_0)}{\partial \theta^3}\right] - 3\mathbb{E}\left[\frac{\partial \ell(\theta_0)}{\partial \theta}\frac{\partial^2 \ell(\theta_0)}{\partial \theta^2}\right]$$

- ...

The first identity (i.e., the mean zero property of the score, i.e., the first derivative of the log-likelihood with respect to parameters) is the population first-order condition for likelihood maximization. It implies, e.g., the

well-known result that maximum likelihood is a generalized method of moment on the score. By the second identity, the second moment of the score is the expectation of the first derivative of the score taken with a minus sign (i.e., Fisher's information matrix). This identity is typically used to simplify the sandwich form of the asymptotic variance of the likelihood estimator. The third identity is a statement about the third moment of the score. It is used in asymptotic refinements and misspecification tests. Naturally, the identities become progressively less familiar as their order increases.

Turning to the approach in Mykland (1994), consider a continuous $\mathcal{F}_t$-semimartingale $X = A + M$, where A is the finite variation component and M is the martingale component. $[M]$ denotes quadratic variation. Write $\Delta_{j+1}M \doteq M_{j+1} - M_j$, with $\Delta_{j+1}[M]$ defined similarly.

- The first identity:

$$\mathbb{E}\left[\Delta_{j+1}M \,\middle|\, \mathcal{F}_j\right] = 0 \tag{1}$$

- The second identity:

$$\mathbb{E}\left[\left(\Delta_{j+1}M\right)^2 \,\middle|\, \mathcal{F}_j\right] = \mathbb{E}\left[\Delta_{j+1}[M] \,\middle|\, \mathcal{F}_j\right] \tag{2}$$

- The third identity:

$$\mathbb{E}\left[\left(\Delta_{j+1}M\right)^3 \,\middle|\, \mathcal{F}_j\right] = 3\mathbb{E}\left[\Delta_{j+1}[M]\Delta_{j+1}M \,\middle|\, \mathcal{F}_j\right] \tag{3}$$

- ...

The first Bartlett-type identity is an obvious restatement of the martingale property. The second relates the conditional expectation of squared increments of the martingale M to the conditional expectation of increments of its quadratic variation. This identity is familiar to contributors and readers of the literature we address. The third identity may, instead, not be obvious. Higher-order identities are even less obvious. In essence, relative to the classical Bartlett identities, $\Delta_{j+1}M$ replaces $\frac{\partial \ell(\theta_0)}{\partial \theta}$ and $\Delta_{j+1}[M]$ replaces $-\frac{\partial^2 \ell(\theta_0)}{\partial \theta^2}$.

Mykland's Bartlett-type identities for martingales will be shown to assist the computation of joint moments (of all orders) of (increments of) a semimartingale price process and random durations between price updates when the durations are endogenously determined. In this sense, they should be viewed as helpful tools in the derivation of limiting arguments under sampling endogeneity. Next, we present an (almost) elementary way to derive them for all orders.

Let $H_n(x)$ be the n-th Hermite polynomial, i.e.,

$$H_n(x) \doteq (-1)^n \, e^{x^2/2} \frac{d^n}{dx^n} e^{-x^2/2}.$$

Write the n-th *re-scaled* Hermite polynomial as follows:

$$H_n(a, x) \doteq a^{n/2} H_n\left(\frac{x}{\sqrt{a}}\right).$$

We have the following.

Theorem 2.1. *Let M be a continuous $\mathcal{F}_t$-martingale. For every $n \geq 1$, the n-th Bartlett-type identity is*

$$\mathbb{E}\left[H_n\left(\Delta_{j+1}[M], \Delta_{j+1}M\right) \,\middle|\, \mathcal{F}_j\right] = 0. \tag{4}$$

The Bartlett-type identity of order n is the conditional expectation of the re-scaled Hermite polynomial of order n evaluated at increments of the quadratic variation and increments of the martingale itself. Implementation of Eq. (4) with $n = 1, 2, 3$ delivers Eq. (1), Eq. (2) and Eq, (3). More generally, Eq. (4) yields identities of all orders (c.f. Table 1).

The logic of the result is straightforward. It is well-understood that, if $M = W$, where W is Brownian motion, $H_n(t, W_t)$ is an $\mathcal{F}_t$-martingale, $\forall n$. More generally, it is known that, if X is the semimartingale $A + M$, $H_n([M]_t, M_t)$

Martingale: $H_n([M]_j, M_j)$	**Bartlett-type identity:** $\mathbb{E}_j[H_n(\Delta_{j+1}[M], \Delta_{j+1}M)] = 0$
M_j	$\mathbb{E}_j[\Delta_{j+1}M] = 0$
$M_j^2 - [M]_j$	$\mathbb{E}_j[(\Delta_{j+1}M)^2] = \mathbb{E}_j[\Delta_{j+1}[M]]$
$M_j^3 - 3\,[M]_j\,M_j$	$\mathbb{E}_j[(\Delta_{j+1}M)^3] = 3\,\mathbb{E}_j[\Delta_{j+1}[M]\,\Delta_{j+1}M]$
$M_j^4 - 6\,[M]_j\,M_j^2 + 3\,[M]_j^2$	$\mathbb{E}_j[(\Delta_{j+1}M)^4] = 6\,\mathbb{E}_j[\Delta_{j+1}[M]\,(\Delta_{j+1}M)^2] - 3\,\mathbb{E}_j[(\Delta_{j+1}[M])^2]$
$M_j^5 - 10\,[M]_j\,M_j^3 +$ $+ 15\,[M]_j^2\,M_j$	$\mathbb{E}_j[(\Delta_{j+1}M)^5] = 10\,\mathbb{E}_j[\Delta_{j+1}[M]\,(\Delta_{j+1}M)^3] - 15\,\mathbb{E}_j[(\Delta_{j+1}[M])^2\,\Delta_{j+1}M]$
$M_j^6 - 15\,[M]_j\,M_j^4 +$ $+ 45\,[M]_j^2 M_j^2 - 15\,[M]_j^3$	$\mathbb{E}_j[(\Delta_{j+1}M)^6] = 15\,\mathbb{E}_j[\Delta_{j+1}[M]\,(\Delta_{j+1}M)^4] - 45\,\mathbb{E}_j[(\Delta_{j+1}[M])^2\,(\Delta_{j+1}M)^2]$ $+ 15\,\mathbb{E}_j[(\Delta_{j+1}[M])^3]$

Table 1: The cases $n = 1$ through $n = 6$. We write $\mathbb{E}_j[\cdot] \doteq \mathbb{E}[\cdot \mid \mathcal{F}_j]$.

is - also - an $\mathcal{F}_t$-martingale, $\forall n$. Both are cases of *polynomial* or *Hermite* martingales. Thus, by the martingale property,

$$\mathbb{E}\left[H_n\left([M]_{j+1}, M_{j+1}\right) \,\middle|\, \mathcal{F}_j\right] = H_n\left([M]_j, M_j\right),$$

which, immediately, implies

$$\mathbb{E}\left[\Delta_{j+1}H_n\left([M], M\right) \,\middle|\, \mathcal{F}_j\right] = 0. \tag{5}$$

Crucially (for the argument in Theorem 2.1), the conditional expectation of increments of the polynomial is the same as the conditional expectation of the polynomial of the increments in spite of the nonlinearity of H_n for $n > 1$. Thus, given Eq. (5), we have

$$\mathbb{E}\left[H_n\left(\Delta_{j+1}[M], \Delta_{j+1}M\right) \,\middle|\, \mathcal{F}_j\right] = 0.$$

The latter step is the result of simple martingale arguments (spelled out in the Appendix) and the properties of the Hermite polynomials. Direct verification now confirms that Eq. (4) yields the corresponding identity for each order n.

Remark 1. For clarity, the increments in Theorem 2.1 are taken over a fixed grid. By the *optional stopping theorem*, however, the identities also apply to *stochastic* and *endogenous* intervals. If the time τ_2 is "essentially finite," by the martingale property of the polynomials,

$$\mathbb{E}\left[H_n\left([M]_{\tau_2}, M_{\tau_2}\right) \,\middle|\, \mathcal{F}_{\tau_1}\right] = H_n\left([M]_{\tau_1\wedge\tau_2}, M_{\tau_1\wedge\tau_2}\right),$$

which immediately yields

$$\mathbb{E}\left[\Delta_{\tau_1}H_n\left([M], M\right) \,\middle|\, \mathcal{F}_{\tau_1}\right] = 0,$$

with Δ_{τ_1} defining an increment between τ_1 and $\tau_2 > \tau_1$. As earlier, $\forall n$, the conditional expectation of the increment of the polynomial can be shown to be the conditional expectation of the polynomial of the increment(s):

$$\mathbb{E}\left[H_n\left(\Delta_{\tau_1}[M], \Delta_{\tau_1}M\right) \,\middle|\, \mathcal{F}_{\tau_1}\right] = 0.$$

Remark 2. Limiting arguments in high-frequency econometrics are based on "discretization". When the price process is a continuous semimartingale driven by Brownian shocks, "discretization" results in increments of Brownian motion ($\Delta M = \Delta W$). Increments of the quadratic variation of the underlying martingale are, therefore, time increments ($\Delta[M] = \Delta t$). Computing joint moments of durations and shocks to the underlying randomness, therefore, amounts to computing Bartlett-type identities of alternative orders for $M = W$ (Brownian identities, in what follows). As an example, Theorem 3.1 discusses a stable law for the tricity in which sampling is conducted based on possibly asymmetric hitting times for W. The proof of the theorem uses extensively the first five Brownian identities. Because of the importance of the Brownian identities for asymptotic arguments, below we turn to some of their specificities.

2.1 On polynomial Brownian martingales and chaotic representations

Let $(\Omega, \mathcal{F}_t, \mathbb{P})$ be a probability space with a Wiener process W_t. The space $L^2(\Omega, \mathcal{F}_t, \mathbb{P})$ decomposes into orthogonal sub-spaces (generally called Wiener chaos) which are linearly spanned by iterated Wiener-Itô integrals:

$$L^2 = \bigoplus_{j=0}^{\infty} \mathcal{H}_j.$$

We note that, $\forall n$, $H_n(t, W_t)$ has a representation (due to Itô, 1951) in terms of iterated Wiener-Itô integrals, i.e.,

$$H_n(t, W_t) = n! I_n(1),$$

where $I_n(1) = n! \int_0^t \int_0^{x_n} ... \int_0^{x_2} dW_{x_1} ... dW_{x_{n-1}} dW_{x_n}$. In addition, $\forall n$, $H_n(t, W_t)$ lies in the n-th Wiener chaos. The polynomials $H_k(t, W_t)$ and $H_\ell(t, W_t)$ are, therefore, orthogonal $\forall k \neq \ell$.

Keeping this orthogonality in mind, we re-write the first four Brownian martingales in the first column of Table 1 as follows:

$$\begin{aligned} W_t &= H_1(t, W_t), \quad (6) \\ W_t^2 &= H_2(t, W_t) + t, \quad (7) \\ W_t^3 &= H_3(t, W_t) + 3tW_t, \end{aligned}$$

which implies, using Eq. (6),

$$W_t^3 = H_3(t, W_t) + 3tH_1(t, W_t) \quad (8)$$

and

$$W_t^4 = H_4(t, W_t) + 6tW_t^2 - 3t^2, \quad (9)$$

which implies, using Eq. (7),

$$W_t^4 = H_4(t, W_t) + 6tH_2(t, W_t) + 3t^2. \quad (10)$$

Eqs. (6), (7), (8) and (10) are orthogonal representations of the first four powers of Brownian motion. Identical arguments may, of course, be applied to higher-order powers. These orthogonal representations are sub-cases of the celebrated Wiener-Itô chaos representation according to which any random variable $X \in L^2(\Omega, \mathcal{F}_t, \mathbb{P})$ satisfies

$$X = \mathbb{E}[X] + \sum_{n=0}^{\infty} I_n(f_n), \quad (11)$$

for a suitable deterministic symmetric kernel f_n. In this sense, the Brownian identities (of all orders) are as fundamental as *chaotic representations* of powers of Brownian motion (of all orders). They can be obtained by "inverting" the latter representations (before taking increments and conditional expectations).

3 Limit theory

We begin by specifying the dynamics of the logarithmic price process X and the filtration on which stable convergence results will be proven. Our emphasis will be on the role of alternative information layers and how those layers translate into limiting properties.

Assumption 1. (The DGP for X). *Assume X evolves as*

$$X_t = X_0 + \int_0^t \mu_s \, ds + \int_0^t \sigma_s dW_s,$$

where the volatility process is strictly positive a.s. and satisfies the equation

$$\sigma_t = \sigma_0 + \int_0^t \mu_s' \, ds + \int_0^t \sigma_s' dW_s + \int_0^t \nu_s dV_s,$$

with W and V independent Brownian motions, μ, σ' and ν adapted càdlàg bounded processes, and μ' adapted càdlàg locally bounded process with $\mu'_t \geq -C(1+\sigma_t)$ for a constant $C > 0$.[7]

The process is a continuous semi-martingale with coefficients which may be semi-martingales themselves. As is customary in the literature, we only specify the dynamics of the coefficient (σ_t) playing a role in the limit. The usual representation, however, immediately implies a form of *chaotic representation* for the underlying process. Plugging σ_t into X_t, in fact, we obtain

$$\begin{aligned} X_t &= X_0 + \int_0^t \mu_s\, ds + \int_0^t \sigma_s dW_s \\ &= X_0 + \int_0^t \mu_s\, ds + \int_0^t \left(\sigma_0 + \int_0^s \mu'_u\, du + \int_0^s \sigma'_u dW_u + \int_0^s \nu_u dV_u\right) dW_s \\ &= X_0 + \sigma_0 W_t + \int_0^t \mu_s\, ds + \int_0^t \int_0^s \mu'_u\, du dW_s + \int_0^t \int_0^s \sigma'_u dW_u dW_s + \int_0^t \int_0^s \nu_u dV_u dW_s. \end{aligned}$$

The Brownian motions W and V represent layers of information, layer 0 being associated with W and layer 1 being associated with V. More generally, assuming Brownian semi-martingale dynamics for the drift term μ_t, an additional layer-1 Brownian motion would arise (under the assumption that the drift is not perfectly correlated with the price process). Similarly, the coefficients associated with μ_t (the drift of the drift and the volatility of the drift) and σ_t (the drift of the volatility and the volatility of volatility) would, in general, be associated with layer-2 Brownian motions and so on, in a cascading fashion.

The Brownian filtration we adopt, defined below and used, e.g., in Li et al. (2013) and Potiron and Mykland (2017), is coherent with this logic with the understanding that the process is generated by a finite number of Brownian motions, i.e., a cascade that would eventually end.

Assumption 2. (The filtration). *The filtration $\{\mathcal{F}_t\}_{t\geq 0}$, to which all stochastic elements are adapted, is generated by W, V and* p *additional independent Brownian motions, that is*

$$\mathcal{F}_t = \sigma\left(\underbrace{W_s}_{\text{layer 0}}, \underbrace{V_s}_{\text{layer 1}}, \underbrace{W_s^{(1)}, \ldots, W_s^{(\mathrm{p})}}_{\text{layers } \geq 1}; 0 \leq s \leq t\right), \quad t \geq 0.$$

We now turn to four types of endogenous sampling. We work with hitting times based on a variety of quantities (e.g., W, X, σ and *integrated* variance). We show that hitting times with respect to W and X have *chaotic representation* with a term affected by the layer-0 Brownian motion W. This term results in *strong* endogeneity (defined, earlier, as a *non-zero limit* for the *tricity*). In the presence of leverage effects, hitting times with respect to σ also have a *chaotic representation* with a term affected by the layer-0 Brownian motion W, which again yields *strong* endogeneity. Absence of leverage would, however, make the corresponding durations a function of the layer-1 Brownian motion V *to the first order* and result in *weak* endogeneity (defined, earlier, as an impact on the asymptotic variance of the *tricity* with a *zero limit*). Hitting times with respect to *integrated* variance translate into durations in which both the layer-0 Brownian motion W and the layer-1 Brownian motion V appear, but at a lower stochastic order, which results, again, in *weak* endogeneity. Table 2 in Subsection 3.5 provides a taxonomy.

We conclude this section with notation adopted throughout the paper. For any triangular array of sampling times $\{\tau_{j,n}\}$, we write

$$\Delta_{j,n} \doteq \tau_{j,n} - \tau_{j-1,n}.$$

Similarly, for any process U,

$$\Delta_j^n U \doteq U_{\tau_{j,n}} - U_{\tau_{j-1,n}}, \qquad j \geq 1.$$

For each $j \geq 1$, we also define

$$\delta_j^n U(t) \doteq U_t - U_{\tau_{j-1,n}}, \qquad t \geq \tau_{j-1,n}.$$

Of course, $\delta_j^n U(\tau_{j,n}) = \Delta_j^n U$. Without a time argument, $\delta_j^n U$ denotes the corresponding increment process. Furthermore, $\mathbb{E}_{\tau_{j-1,n}}[\cdot]$ defines $\mathbb{E}[\cdot \mid \mathcal{F}_{\tau_{j-1,n}}]$. For a stopping time ρ and an event A, we write $\mathbb{E}_\rho[U; A] \doteq$

[7] The two hypotheses on μ' (local boundedness and the stated inequality) are sufficient to guarantee that two conditions *i)* $\sigma_t > 0$ (almost surely for all t) and *ii)* $\sigma_t'^2 + \nu_t^2 \geq c_0 > 0$ (almost surely for all t and a constant $c_0 > 0$) are not mutually exclusive. The second (i.e. the existence of a "deterministic floor" for the instantaneous variance of σ), in particular, is sufficient to make sure that the sampling times used in Theorem 3.4 are regular enough to satisfy the hypotheses of Doob's optional stopping theorem. As a possible example, we may have $d\sigma_t = (\mu_0/\sigma_t)\, dt + \gamma\, dW_t + \rho\, dV_t$, for suitable constants μ_0, γ and ρ.

$\mathbb{E}[U\,\mathbb{1}_{\{A\}} \mid \mathcal{F}_\rho]$.

3.1 Hitting times of an asymmetric barrier for W

In this first example, we assume the observation times are the result of the Brownian motion W hitting asymmetric barriers given by a and b, with $a > b$. This is Example 4 in Li et al. (2013) and a version of Example 6 in Potiron and Mykland (2017), one in which the hitting Brownian motion is - by definition - perfectly correlated with W. Consistent with the duration modeling in Renault et al. (2014), Potiron and Mykland (2017) refer to this observation scheme as being *structural*.

Assumption 3. (Hitting times of an asymmetric barrier for W.) *Let $a > 0$ and $b > 0$ be positive real numbers. For all $n \in \mathbb{N}$, $n > 0$, consider the real open interval $\mathcal{U}_n \doteq (-n^{-1/2}\, b, n^{-1/2}\, a)$. We assume that the observation times $\{\tau_{j,n}\}$ are recursively generated by*

$$\begin{cases} \tau_{0,n} = 0 \\ \tau_{j+1,n} = \inf\left\{ t \geq \tau_{j,n} \;\middle|\; W_t - W_{\tau_{j,n}} \notin \mathcal{U}_n \right\}. \end{cases} \tag{12}$$

Assumption 3 implies the following representation for the durations between price updates.

Lemma 1. (A chaotic representation of the durations.) *Given Assumption 3, we have*

$$\tau_{j,n} - \tau_{j-1,n} = \frac{ab}{n} + \int_{\tau_{j-1,n}}^{\tau_{j,n}} \Phi_{j,n}^{(W)}(t)\; dW_t, \tag{13}$$

where, for $j \geq 1$,

$$\Phi_{j,n}^{(W)}(t) \doteq ((a-b)/\sqrt{n} - 2\,\delta_j^n W(t)) = O_p(n^{-1/2}),$$

uniformly on $[\tau_{j-1,n}, \tau_{j,n}]$.

The implied durations are W-semi-martingales with conditional mean ab/n and martingale component

$$\int_{\tau_{j-1,n}}^{\tau_{j,n}} \Phi_{j,n}^{(W)}(t)\; dW_t.$$

The former vanishes (deterministically) at rate n. The second vanishes (in probability) at rate n.

Theorem 3.1. *Let Assumptions 1, 2 and 3 hold. As $n \to \infty$, we have the following stable law:*

$$\begin{aligned} n^{1/2}\left(\sum_{\tau_{j,n} \leq t} n^{1/2} \left(\Delta_j^n X\right)^3 - (a-b)\int_0^t \sigma_s^3\, ds \right) \xrightarrow{\mathcal{L}-s} \; & c_0(a,b) \int_0^t \sigma_s^2\, \mu_s\, ds + c_1(a,b) \int_0^t \sigma_s^2\, \sigma_s'\, ds + \\ & c_2(a,b) \int_0^t \sigma_s^3\, dW_s + c_3(a,b) \int_0^t \sigma_s^3\, dW_s', \end{aligned} \tag{14}$$

where W' is a Brownian motion defined on an extension of the original probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ and independent of the σ-algebra $\mathcal{F}$ and the coefficients $c_k(a,b)$, with $k = 0, \cdots, 3$, are given by

$$\begin{aligned} c_0(a,b) &\doteq \frac{1}{3}\left(a^2 + a\,b + b^2\right), \\ c_1(a,b) &\doteq \frac{a\,b}{2}, \\ c_2(a,b) &\doteq \frac{1}{3}\left(2a^2 - ab + 2b^2\right), \\ c_3(a,b) &\doteq \left(\frac{2}{9}(a-b)^2\left(a^2 + ab + b^2\right)\right)^{1/2}. \end{aligned}$$

Remark 3. Theorem 3.1 extends to a specific form of endogenous sampling the deterministic equi-spaced case discussed in Kinnebrock and Podolskij (2008) (Example 6) and the exogenous random case in Li et al. (2013) (Theorem 2). If the barriers are genuinely asymmetric ($a \neq b$), the *tricity* is asymptotically biased with an inflated variance. The general case with $a \neq b$ is, therefore, a case of *strong endogeneity*. Should the barriers be symmetric, then the bias would be zero but the limiting distribution would continue to be uncentered and, in general, not mixed normal.

Remark 4. The proof of the theorem, which consists of the verification of the five conditions in Jacod (2008), hinges heavily on the use of Bartlett-type identities applied to the driving Brownian motion W. The conditions typically involve cross-moments (with various powers) of the random durations and W, which is what the Bartlett-type identities provide (c.f. Table 1). The same observation will apply to alternative observation schemes (discussed below) and will not be repeated.

Remark 5. We will later show that *chaotic representations* of the durations are particularly helpful to prove one of the conditions for the stable law in Jacod (2008), i.e., the orthogonality condition (relative to all bounded $\mathcal{F}_t$-martingales N_t such that $[N, W]_t = 0$) for a limiting representation on a good extension of the original probability space. Contrary to other examples of endogenous sampling described below, however, Jacod's orthogonality condition is immediately satisfied for this observation scheme. Thus, the *chaotic representation* of the durations in Lemma 1 may *not* be used in the proof of the theorem. The logic as to why the condition is immediately satisfied in this case is as follows. Write the Kunita-Watanabe representation of N_t as

$$N_t = \sum_{k=0}^{\mathrm{p}} \int_0^t \eta_s^{(k)} \, dW_s^{(k)},$$

where $W^{(0)} \equiv V$ and the $\{\eta_t^{(k)}\}$s are suitable processes. Take any $j \geq 1$ and consider the σ-algebra

$$\mathcal{H} \doteq \mathcal{F}_{\tau_{j-1,n}} \vee \sigma\left\{ W_{\tau_{j-1,n}+s} - W_{\tau_{j-1,n}} \;\middle|\; s \geq 0 \right\}.$$

Naturally, $\mathcal{F}_{\tau_{j-1,n}} \subset \mathcal{H}$. We have that $\tau_{j-1,n}$ and $\tau_{j,n}$ are both $\mathcal{H}$-measurable and so is the duration $\Delta_{j,n} = \tau_{j,n} - \tau_{j-1,n} \in \mathcal{H}$. Also,

$$\mathbb{E}\left[\int_{\tau_{j-1,n}}^{\tau_{j,n}} \eta_s^{(k)} \, dW_s^{(k)} \;\middle|\; \mathcal{H} \right] = 0, \quad k = 0, \cdots, \mathrm{p}.$$

Accordingly, the $\mathcal{F}_{\tau_{j-1,n}}$-conditional expected value of the product between the j-th time increment and the corresponding N increment (i.e., the central element of Jacod's orthogonality condition in the case of endogenous sampling) is

$$\mathbb{E}\left[\Delta_{j,n} \Delta_j^n N \;\middle|\; \mathcal{F}_{\tau_{j-1,n}} \right] = \mathbb{E}\left[\mathbb{E}\left[\Delta_{j,n} \Delta_j^n N \;\middle|\; \mathcal{H} \right] \;\middle|\; \mathcal{F}_{\tau_{j-1,n}} \right] = \sum_{k=0}^{\mathrm{p}} \mathbb{E}\left[\Delta_{j,n} \, \mathbb{E}\left[\int_{\tau_{j-1,n}}^{\tau_{j,n}} \eta_s^{(k)} \, dW_s^{(k)} \;\middle|\; \mathcal{H} \right] \;\middle|\; \mathcal{F}_{\tau_{j-1,n}} \right] = 0.$$

In essence, the orthogonality condition (or, equivalently, Eq. (10) in Podolskij and Vetter, 2010) is satisfied for every n and not just asymptotically. As mentioned, this result is specific to the adopted observation scheme and is, therefore, not general.

Remark 6. We note that, if sampling were the objective as in Li et al. (2025), the type of hitting time sampling in this subsection would be econometrically *infeasible*. In this sense, modeling hitting times relative to W should only be viewed as (1) either a primitive *structural* assumption on the generation of observations or (2) as a preliminary case on which richer observation/sampling schemes will logically build. One such scheme is discussed in the next subsection.

3.2 Hitting times of an asymmetric barrier for X

We now work with possibly asymmetric barriers defined on the full process X. An identical scheme is adopted by Li et al. (2025) to *sample* the process rather than to *model* the observation times.

Assumption 4. (Hitting times of an asymmetric barrier by X.) *Let $a > 0$ and $b > 0$ be positive real numbers. For all $n \in \mathbb{N}$, $n > 0$, consider the real open interval $\mathcal{U}_n \doteq (-n^{-1/2} b, n^{-1/2} a)$. We assume that the observation times*

$\{\tau_{j,n}\}$ are recursively generated by

$$\begin{cases} \tau_{0,n} = 0 \\ \tau_{j+1,n} = \inf\left\{ t \geq \tau_{j,n} \;\middle|\; X_t - X_{\tau_{j,n}} \notin \mathcal{U}_n \right\}. \end{cases} \tag{15}$$

Assume, further, that[8] $\exists c_0 > 0 : \mathbb{P}\left[\sigma_t^2 \geq c_0,\ \forall t \geq 0\right] = 1.$

Assumption 4 implies the following representation for the durations between price updates.

Lemma 2. (A chaotic representation of the durations.) *We have*

$$\tau_{j,n} - \tau_{j-1,n} = \frac{ab}{n\,\sigma^2_{\tau_{j-1,n}}} + \int_{\tau_{j-1,n}}^{\tau_{j,n}} \Phi^{(W)}_{j,n}(t)\; dW_t + \int_{\tau_{j-1,n}}^{\tau_{j,n}} \Phi^{(V)}_{j,n}(t)\; dV_t + \int_{\tau_{j-1,n}}^{\tau_{j,n}} \Phi^{(\Delta)}_{j,n}(t)\; dt \tag{16}$$

where, for $j \geq 1$,

$$\Phi^{(W)}_{j,n}(t) \doteq u^{(n)}_y\,\sigma_t + u^{(n)}_\sigma\,\sigma'_t, \quad \Phi^{(V)}_{j,n}(t) \doteq u^{(n)}_\sigma\,\nu_t, \quad \Phi^{(\Delta)}_{j,n}(t) \doteq u^{(n)}_y\,\mu_t + u^{(n)}_\sigma\,\mu'_t + \frac{1}{2}\,u^{(n)}_{\sigma\sigma}\,\left(\sigma'^2_t + \nu^2_t\right) + u^{(n)}_{y\sigma}\,\sigma_t\,\sigma'_t,$$

and

$$\begin{aligned}
u^{(n)}_y\left(\delta^n_j X(t), \sigma_t\right) &= \frac{1}{\sigma^2_t}\left(\frac{a-b}{\sqrt{n}} - 2\,\delta^n_j X(t)\right) = O_p\left(n^{-1/2}\right), \\
u^{(n)}_\sigma\left(\delta^n_j X(t), \sigma_t\right) &= -\frac{2}{\sigma^3_t}\left(\frac{a\,b}{n} + \frac{(a-b)}{\sqrt{n}}\,\delta^n_j X(t) - \left(\delta^n_j X(t)\right)^2\right) = O_p\left(n^{-1}\right), \\
u^{(n)}_{\sigma\sigma}\left(\delta^n_j X(t), \sigma_t\right) &= \frac{6}{\sigma^4_t}\left(\frac{a}{\sqrt{n}} - \delta^n_j X(t)\right)\left(\frac{b}{\sqrt{n}} + \delta^n_j X(t)\right) = O_p\left(n^{-1}\right), \\
u^{(n)}_{y\sigma}\left(\delta^n_j X(t), \sigma_t\right) &= -\frac{2}{\sigma^3_t}\left(\frac{a-b}{\sqrt{n}} - 2\,\delta^n_j X(t)\right) = O_p\left(n^{-1/2}\right),
\end{aligned}$$

uniformly on $[\tau_{j-1,n}, \tau_{j,n}]$. The probability orders derive from the fact that $\delta^n_j X(t) = O_p(n^{-1/2})$, uniformly on $[\tau_{j-1,n}, \tau_{j,n}]$. As a consequence,

$$\Phi^{(W)}_{j,n}(t) = O_p\left(n^{-1/2}\right), \;\; \Phi^{(V)}_{j,n}(t) = O_p\left(n^{-1}\right) \;\; \textit{and} \;\; \Phi^{(\Delta)}_{j,n}(t) = O_p\left(n^{-1/2}\right).$$

The implied durations are now $\{W, V\}$-semi-martingales with conditional mean

$$\frac{ab}{n\,\sigma^2_{\tau_{j-1,n}}} + \mathbb{E}_{\tau_{j-1,n}}\left[\int_{\tau_{j-1,n}}^{\tau_{j,n}} \Phi^{(\Delta)}_{j,n}(t)\; dt\right].$$

Importantly, the asymptotic order of the second martingale component (the component driven by V) is lower ($O_p(1/n^{3/2})$) than that of the first martingale component ($O_p(1/n)$). The intuition for this result - which is central to our limiting arguments - is contained in the following remark.

Remark 7. Assume our original filtration generated by a finite number ($K = \mathrm{p} + 2$) of Brownian motions. In order to lay out the logic without unnecessary notation, simplify the price process to

$$\begin{aligned}
dX_t &= \mu_t dt + \sigma_t dW_t, \\
d\sigma_t &= \mu'_t dt + \nu_t dV_t.
\end{aligned}$$

By the Clark-Ocone theorem, for every square-integrable random variable (with suitable smoothness conditions)

[8] The requirement of a strictly positive lower bound for σ guarantees that sampling through hitting times of a spatial barrier for X generates stopping times that satisfy the sufficient conditions of Doob's optional stopping theorem. Analogous assumptions will be made in subsequent theorems with the same objective. This comment will, therefore, not be repeated.

on the above filtration, we may write:

$$F_T = \mathbb{E}[F_T] + \sum_{k=1}^{K} \int_0^T \mathbb{E}\left[D_t^{W^k} F_T \,\middle|\, \mathcal{F}_t \right] dW_t^k,$$

where the quantities $\mathbb{E}[D_t^{W^k} F_T \mid \mathcal{F}_t]$ are expectations of Malliavin derivatives. Assume now $F_T = h(X_T)$ for some differentiable function $h(X_T)$. We have

$$D_t^{W^k} F_T = h'(X_T) D_t^{W^k} X_T.$$

Focusing on the first two Brownian motions (W and V), we have

$$D_t^W X_T = D_t^W \left(\int_0^T \mu_s ds \right) + D_t^W \left(\int_0^T \sigma_s dW_s \right),$$

which gives $D_t^W X_T = \sigma_t$. Turning to $D_t^V X_T$, we obtain

$$\begin{aligned} D_t^V X_T &= D_t^V \left(\int_0^T \mu_s ds \right) + D_t^V \left(\int_0^T \sigma_s dW_s \right) = \int_0^T D_t^V \sigma_s dW_s \\ &= \int_0^T \left(D_t^V \int_0^s \nu_u dV_u \right) dW_s \\ &= \nu_t (W_T - W_t). \end{aligned}$$

Therefore, the sensitivity with respect to the Brownian motion V is $1/\sqrt{n}$ smaller than the one with respect to the Brownian motion W. In essence, lower-layer shocks affect the process less than higher-layer shocks. The lower the layer, the more muted the impact.

Remark 8. We note that the *chaotic representation* in Eq. (16) is defined *implicitly* in terms of (*i*) layers of information (as given by the Brownian motions W and V) and (*ii*) stochastic orders (associated with the terms $\Phi_{j,n}^{(W)}(t)$, $\Phi_{j,n}^{(V)}(t)$, and $\Phi_{j,n}^{(\Delta)}(t)$). To derive stable limits (our objective in this article), one does not need to go deeper into the *chaotic representation* and write it - as typically done - in terms of *iterated* integrals (c.f. Eq. 11). Consider, as an example, the term $\int_{\tau_{j-1,n}}^{\tau_{j,n}} \Phi_{j,n}^{(W)}(t)\, dW_t$. Because $\Phi_{j,n}^{(W)}(t)$ is a function of $\delta_j^n X(t) \doteq X_t - X_{\tau_{j-1,n}}$, and $\delta_j^n X(t)$ has an integral representation, we could easily iterate integrals. We do not do it here because it is unnecessary. In this sense, the use of the terminology "chaos" in the derived representations for $\tau_{j,n} - \tau_{j-1,n}$ is consistent with common practice. More specifically, our emphasis on information layers is at the core of chaos theory. The same considerations apply to the expansions below and will not be repeated.

Theorem 3.2. *Let Assumptions 1, 2 and 4 hold. As $n \to \infty$, we have the following stable law:*

$$\begin{aligned} & n^{1/2} \left(\sum_{\tau_{j,n} \le t} n^{1/2} \left(X_{\tau_{j,n}} - X_{\tau_{j-1,n}} \right)^3 - (a-b) \int_0^t \sigma_s^2 \, ds \right) \\ & \xrightarrow{\mathcal{L}-s} -\frac{2}{3} (a-b)^2 \int_0^t \sigma_s' \, ds \; + \; d_0(a,b) \int_0^t dX_s \; + \; d_1(a,b) \int_0^t \sigma_s \, dW_s', \end{aligned}$$

where W' is a Brownian motion defined on an extension of the original probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \ge 0}, \mathbb{P})$ and independent of the σ-algebra $\mathcal{F}$ and the coefficients $d_0(a,b)$ and $d_1(a,b)$ are defined as

$$d_0(a,b) \doteq \frac{2a^2 - ab + 2b^2}{3}, \tag{17}$$

$$d_1(a,b) \doteq (a-b) \left(\frac{2}{9} \left(a^2 + ab + b^2 \right) \right)^{1/2}. \tag{18}$$

Remark 9. The different source of endogeneity (relative to the W case) leads to a different stable law, both in terms of asymptotic centering and in terms of asymptotic variance. The limiting bias, in particular, is now $(a-b) \int_0^t \sigma_s^2 \, ds$, i.e., a multiple of integrated variance rather than a multiple of the integral of third powers of σ, as in Theorem 3.1. The different scaling depends on the scaling (by powers of σ) of the duration expansion in Eq.

(16). The first term, for example, is $ab/(n\,\sigma^2_{\tau_{j-1,n}})$. For comparison, the corresponding term in Eq. (13) is ab/n. Thus, interestingly, in the general case $a \neq b$, one could estimate *integrated variance* by using a properly-rescaled third-moment estimator. Having made this point, the general case $a \neq b$ leads to *strong endogeneity*, like in Subsection 3.1. If $a = b$, the bias is zero and the asymptotic variance is minimized.

Remark 10. Because of the presence of the Brownian motion V in the representation of the durations, Eq. (10) in Podolskij and Vetter (2010) is now *not* satisfied for every n. While the argument in Remark 5 continues to apply to the first two terms in the duration representation in Eq. (16) for every n, it cannot apply to the third term (i.e., the integral with respect to V). The condition is, however, satisfied asymptotically because of the (sufficiently small) asymptotic order of $u^{(n)}_{\sigma}(\delta^n_j X(t), \sigma_t)\,\nu_t$, a consequence of the layer-1 nature of volatility, as we explained in Remark 7. We note that the fourth term in Eq. (16) is of the same probability order as the third term, i.e. $O_p(n^{-3/2})$, and can, therefore, be treated similarly.

3.3 Hitting times of an asymmetric barrier for *integrated* variance

Next, we work with the random clock $\mathrm{IV}_t \doteq \int_0^t \sigma_s^2\,ds$. We assume the observations are generated every time integrated variance reaches multiples of a threshold a/n. Economic justifications for a random clock derive from the classical work of Clark (1973) and Tauchen and Pitts (1983).

Assumption 5. (Hitting times of a barrier for IV.) *Let $a > 0$ be positive real numbers. For all $n \in \mathbb{N}$, $n > 0$, we assume that the observation times $\{\tau_{j,n}\}$ are recursively generated by*

$$\begin{cases} \tau_{0,n} &= 0, \\ \tau_{j,n} &= \inf\left\{t \geq \tau_{j-1,n} \,:\, IV_t - IV_{\tau_{j-1,n}} \geq a/n\right\}. \end{cases}$$

We assume, further, that $\exists c_{IV} > 0 : \mathbb{P}\left[\sigma_t'^2 + \nu_t^2 > c_{IV},\ \forall t \geq 0\right] = 1$.

Assumption 5 implies the following representation for the durations between price updates.

Lemma 3. (A chaotic representation of the durations.) *We have*

$$\tau_{j,n} - \tau_{j-1,n} = \frac{a}{n\sigma^2_{\tau_{j-1,n}}} + \int_{\tau_{j-1,n}}^{\tau_{j,n}} \Phi^{(W)}_{j,n}(t)\,dW_t + \int_{\tau_{j-1,n}}^{\tau_{j,n}} \Phi^{(V)}_{j,n}(t)\,dV_t + \int_{\tau_{j-1,n}}^{\tau_{j,n}} \Phi^{(\Delta)}_{j,n}(t)\,dt, \tag{19}$$

where, for $j \geq 1$,

$$\begin{aligned} \Phi^{(W)}_{j,n}(t) &\doteq -2\left(\frac{a}{n} - \delta^n_j IV(t)\right)\frac{\sigma_t'}{\sigma_t^3} = O_p\left(n^{-1}\right) \\ \Phi^{(V)}_{j,n}(t) &= -2\left(\frac{a}{n} - \delta^n_j IV(t)\right)\frac{\nu_t}{\sigma_t^3} = O_p\left(n^{-1}\right), \\ \Phi^{(\Delta)}_{j,n}(t) &= -2\left(\frac{a}{n} - \delta^n_j IV(t)\right)\frac{\mu_t'}{\sigma_t^3} + 3\left(\frac{a}{n} - \delta^n_j IV(t)\right)\frac{\sigma_t'^2 + \nu_t^2}{\sigma_t^4} = O_p\left(n^{-1}\right), \end{aligned}$$

uniformly on $[\tau_{j-1,n}, \tau_{j,n}]$.

The Brownian motions W and V affecting spot variance in layer 1 (as well as the price process directly - in the case of W - in layer 0) enter the expansion with coefficients that have the same probability orders ($O_p(1/n)$). Therefore, both local martingale components in the expansion in Eq. (19) have order $O_p(n^{-3/2})$. This should be contrasted with the case in Subsection 3.2 in which the W-local martingale has an order $O_p(n^{-1})$ which is higher than the order of the V-local martingale, i.e., $O_p(n^{-3/2})$. This difference is consequential, as we discuss in Remark 11. The fourth term in Eq. (19) is of order $O_p(n^{-2})$.

Theorem 3.3. *Let Assumptions 1, 2 and 5 hold. As $n \to \infty$, we have the following stable law:*

$$\sqrt{n} \sum_{\tau_{j,n} \leq t} \sqrt{n} \left(\Delta_j^n X\right)^3 \xrightarrow{\mathcal{L}-s} 3\,a \int_0^t \mu_s\, ds + 3\,a \int_0^t \sigma_s\, dW_s + \sqrt{6}\,a \int_0^t \sigma_s\, dW'_s, \tag{20}$$

where W' is a Brownian motion defined on an extension of the original probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ and independent of the σ-algebra $\mathcal{F}$.

Remark 11. A key implication of Theorem 3.3 is absence of a limiting bias for the *tricity*. The *weak* endogeneity in Theorem 3.3 should, therefore, be compared to the *strong* endogeneity in the previous two cases. We note that, differently from the model in Bandi et al. (2023),[9] a zero limit (and, therefore, *weak* endogeneity) in this case is not the result of durations that only depend on an orthogonal (to the layer-0 Brownian motion W) layer-1 Brownian motion V. It is, instead, the result of the order of the stochastic integrals with respect to both W and V, which derives from defining the barriers with respect to random clocks, i.e., *integrated* variance. If we defined the barriers with respect to *spot* variance, instead, both the layer-0 Brownian motion W and the layer-1 Brownian motion V would play a role, in general, with the *same* stochastic order, thereby re-establishing *strong* endogeneity (unless leverage effects are absent). Below, we study this specific case.

3.4 Hitting times of an asymmetric barrier for *spot* volatility

In Assumption 6, we specify hitting barriers relative to *spot* volatility.

Assumption 6. (Hitting times of a barrier for σ.) *Let $a > 0$ and $b > 0$ be positive real numbers. For all $n \in \mathbb{N}$, $n > 0$, consider the real open interval $\mathcal{U}_n \doteq (-n^{-1/2} b, n^{-1/2} a)$. We assume that the observation times $\{\tau_{j,n}\}$ are recursively generated by*

$$\begin{cases} \tau_{0,n} &= 0, \\ \tau_{j,n} &= \inf\left\{t \geq \tau_{j-1,n} \,:\, \sigma_t - \sigma_{\tau_{j-1,n}} \notin \mathcal{U}_n\right\}. \end{cases}$$

Assume, further, that $\exists c'_0 > 0 : \mathbb{P}\left[\sigma_t'^2 + \nu_t^2 > c'_0,\ \forall t \geq 0\right] = 1$.

To derive a *chaotic representation* for the durations under Assumption 6, we enrich Assumption 1 with Assumption 7. In essence, we add a second layer to the dynamics of the price process.

Assumption 7. *Assume that the processes σ' and ν in the dynamics of the instantaneous volatility σ have the structure*

$$\begin{aligned} d\sigma'_t &= \mu''_t\, dt + \sigma''_t\, dW_t + \nu''_t\, dV_t + \gamma_t\, dZ_t, \\ d\nu_t &= \overline{\mu}_t\, dt + \overline{\sigma}_t\, dW_t + \overline{\nu}_t\, dV_t + \overline{\gamma}_t\, dB_t, \end{aligned}$$

where B and Z are Brownian motions independent of (W, V) and independent of each other.

The role of Assumption 7 is to explicitly model σ', i.e., the volatility of volatility associated with the common (between the X process and the spot volatility process) Brownian motion W and ν, i.e., the volatility of volatility associated with the orthogonal (between the X process and the spot volatility process) Brownian motion V. Ignoring the volatility drift (μ') which does not have to be specified separately, Assumption 7 models the next layer (i.e., the second layer) in the cascade of shocks affecting the process. We allow both σ', which captures traditional leverage effects, and ν to be correlated with both the X process and the spot volatility process, which leads to deeper leverage-type effects through σ'', ν'', $\overline{\sigma}$, and $\overline{\nu}$. The Brownian motions Z and B capture independent variation from shocks in higher layers.

Lemma 4. (A chaotic representation of the durations.) *We have*

$$\Delta_{j,n} = \frac{ab}{n\,\Gamma_{\tau_{j-1,n}}} + \int_{\tau_{j-1,n}}^{\tau_{j,n}} \Phi_{j,n}^{(W)}(t)\, dW_t + \int_{\tau_{j-1,n}}^{\tau_{j,n}} \Phi_{j,n}^{(V)}(t)\, dV_t + \int_{\tau_{j-1,n}}^{\tau_{j,n}} \Phi_{j,n}^{(Z)}(t)\, dZ_t + \int_{\tau_{j-1,n}}^{\tau_{j,n}} \Phi_{j,n}^{(B)}(t)\, dB_t + \int_{\tau_{j-1,n}}^{\tau_{j,n}} \Phi_{j,n}^{(\Delta)}(t)\, dt, \tag{21}$$

[9] In Bandi et al. (2023), dependence between the durations and volatility is generated through the dependence between the time-varying likelihood of price inaction and volatility. The time-varying likelihood of price inaction, however, solely depends on the layer-1 Brownian motion V.

where $\Gamma_t \doteq \sigma_t'^2 + \nu_t^2$ *and, for* $j \geq 1$*, uniformly in* $t \in [\tau_{j-1,n}, \tau_{j,n}]$*, the coefficients of the representation are*

$$\begin{aligned}
\Phi_{j,n}^{(W)}(t) &\doteq u_y\,\sigma_t' + u_s\,\sigma_t'' + u_v\,\overline{\sigma}_t, \quad \Phi_{j,n}^{(V)}(t) \doteq u_y\,\nu_t + u_s\,\nu_t'' + u_v\,\overline{\nu}_t,\\
\Phi_{j,n}^{(Z)}(t) &\doteq u_s\,\gamma_t, \quad \Phi_{j,n}^{(B)}(t) \doteq u_v\,\overline{\gamma}_t,\\
\Phi_{j,n}^{(\Delta)}(t) &\doteq u_y\,\mu_t' + u_s\,\mu_t'' + u_v\,\overline{\mu}_t + \frac{1}{2}u_{ss}\big((\sigma_t'')^2 + (\nu_t'')^2 + \gamma_t^2\big) + \frac{1}{2}u_{vv}\big((\overline{\sigma}_t)^2 +\\
&\quad (\overline{\nu}_t)^2 + (\overline{\gamma}_t)^2\big) + u_{sv}\big(\sigma_t''\overline{\sigma}_t + \nu_t''\overline{\nu}_t\big) + u_{ys}\big(\sigma_t'\sigma_t'' + \nu_t\nu_t''\big) + u_{yv}\big(\sigma_t'\overline{\sigma}_t + \nu_t\overline{\nu}_t\big),
\end{aligned}$$

where the derivatives are evaluated at $(\delta_j^n\sigma(t), \sigma_t', \nu_t)$*, with* $U_n \doteq a/\sqrt{n}$*,* $\ell_n \doteq b/\sqrt{n}$*,*

$$\begin{aligned}
u_y &= \frac{U_n - \ell_n - 2\delta_j^n\sigma(t)}{\Gamma_t} = O_p(n^{-1/2}), \quad u_s = u_{\sigma'} = -\frac{2\sigma_t'\,q_t}{\Gamma_t^2} = O_p(n^{-1}), \quad u_v = u_\nu = -\frac{2\nu_t\,q_t}{\Gamma_t^2} = O_p(n^{-1}),\\
u_{ss} &= -\frac{2q_t}{\Gamma_t^2} + \frac{8\sigma_t'^2\,q_t}{\Gamma_t^3} = O_p(n^{-1}), \qquad u_{vv} = -\frac{2q_t}{\Gamma_t^2} + \frac{8\nu_t^2\,q_t}{\Gamma_t^3} = O_p(n^{-1}), \quad u_{sv} = \frac{8\sigma_t'\nu_t\,q_t}{\Gamma_t^3} = O_p(n^{-1}),\\
u_{ys} &= -\frac{2\sigma_t'\,(U_n - \ell_n - 2\delta_j^n\sigma(t))}{\Gamma_t^2} = O_p(n^{-1/2}), \qquad u_{yv} = -\frac{2\nu_t\,(U_n - \ell_n - 2\delta_j^n\sigma(t))}{\Gamma_t^2} = O_p(n^{-1/2}),
\end{aligned}$$

and $q_t \doteq (U_n - \delta_j^n\sigma(t))(\ell_n + \delta_j^n\sigma(t))$*. The probability orders derive from the fact that* $\delta_j^n\sigma(t) = O_p(n^{-1/2})$ *uniformly on* $[\tau_{j-1,n}, \tau_{j,n}]$*. Accordingly,*

$$\Phi_{j,n}^{(W)}(t) = O_p(n^{-1/2}), \quad \Phi_{j,n}^{(V)}(t) = O_p(n^{-1/2}), \quad \Phi_{j,n}^{(Z)}(t) = O_p(n^{-1}), \quad \Phi_{j,n}^{(B)}(t) = O_p(n^{-1}), \quad \Phi_{j,n}^{(\Delta)}(t) = O_p(n^{-1/2}).$$

Remark 12. V is a layer-1 Brownian motion. So is W (which is also a layer-0 Brownian motion, like in previous model specifications) because of allowance for leverage effects. Without restrictions on the parameters, the corresponding integrals enter the *chaotic representation* to the same order since spot variance is affected both by V and W and the barrier is defined with respect to spot variance. In this sense, this case is analogous to the integrated variance case in Subsection 3.3. This said, because of the spot nature of the current problem, both integrals have a lower probability order than in the integrated case (i.e., n^{-1} vs. $n^{-3/2}$ in the integrated case). The integral with respect to W leads to a non-zero limiting bias for the *tricity*. The result hinges on its interaction with the driving Brownian motion of the price process W, when leverage effects are present, i.e., when σ' is different from zero. Theorem 3.4 formalizes this result.

Remark 13. When leverage effects are absent, i.e., when σ_t' is zero, the durations are affected by W at a lower order (since $\Phi_{j,n}^{(W)}(t) \doteq u_v\,\overline{\sigma}_t$). The order of the corresponding integral is $n^{-3/2}$ rather than n^{-1}. While the durations are *still* affected by W, the lower order is sufficient to lead to a zero limit for the *tricity*. We note that it is not surprising to have spot-volatility-driven durations that are a function of W even though traditional leverage effects are muted ($\sigma' = 0$). The result hinges on the fact that, even though layer 0 of spot volatility does not depend on W, layer 1 (i.e., the second layer of the process X) is still a function of W through ν provided $\overline{\sigma} \neq 0$. In sum, when traditional leverage effects are absent, the integral with respect to W has the same order as the integrals with respect to B and Z. This is because W is, in this case, a layer-1 shock of the spot volatility process (or, equivalently, a layer-2 shock of the X process) just like B and Z.

Theorem 3.4. *Let Assumptions 1, 2, 6, and 7 hold. Call* $\Gamma_t \doteq \sigma_t'^2 + \nu_t^2$*. As* $n \to \infty$*, the following law of large numbers applies:*

$$\sum_{\tau_{j,n} \leq t} \sqrt{n}\left(\Delta_j^n X\right)^3 \xrightarrow{\;p\;} (a-b)\int_0^t \frac{\sigma_s^3}{\Gamma_s}\,\sigma_s'\,ds.$$

Under the (simplifying) assumptions

$$\mu_t'' \equiv \mu_0'', \quad \sigma_t'' \equiv 0, \quad \nu_t'' \equiv 0, \quad \gamma_t \equiv \gamma_0, \quad \overline{\mu}_t \equiv \overline{\mu}_0, \quad \overline{\sigma}_t \equiv 0, \quad \overline{\nu}_t \equiv 0, \quad \overline{\gamma}_t \equiv \overline{\gamma}_0, \tag{22}$$

we also have the stable law

$$\sqrt{n}\left(\sum_{\tau_{j,n} \leq t} \sqrt{n}\left(\Delta_j^n X\right)^3 - (a-b)\int_0^t \frac{\sigma_s^3}{\Gamma_s}\,\sigma_s'\,ds\right) \xrightarrow{\mathcal{L}-s} \mathcal{B}_t + \int_0^t H_s\,dW_s + \int_0^t K_s\,dV_s + \int_0^t \Lambda_s\,dW_s', \tag{23}$$

where W' is a Brownian motion independent of both W and V, the finite-variation process $\mathcal{B}_t$ is defined as

$$\mathcal{B}_t \doteq -\frac{a^2+4\,a\,b+b^2}{3}\int_0^t \frac{\mu_s'\,\sigma_s^3\,\sigma_s'}{\Gamma_s^2}\,ds + (a^2+3ab+b^2)\int_0^t \frac{\mu_s\,\sigma_s^2}{\Gamma_s}\,ds - 2a\,b\int_0^t \frac{\mu_s\,\sigma_s^2\,\sigma_s'^2}{\Gamma_s^2}\,ds, \tag{24}$$

the processes H_t, K_t and Λ_t are

$$H_t \doteq \frac{\sigma_t^3}{3\Gamma_t^2}\Big(3(a^2+3ab+b^2)\Gamma_t - (a^2+10ab+b^2)\sigma_t'^2\Big), \tag{25}$$

$$K_t \doteq -\frac{(a^2+4ab+b^2)\sigma_t^3\sigma_t'\nu_t}{3\Gamma_t^2}, \tag{26}$$

$$\Lambda_t \doteq \sqrt{\frac{\sigma_t^6}{15\Gamma_t^2}q_{a,b}\left(\frac{\sigma_t'^2}{\Gamma_t}\right) - H_t^2 - K_t^2}, \tag{27}$$

the function $q_{a,b}(\cdot)$ is

$$q_{a,b}(x) \doteq e_2(a,b)\,x^2 - e_1(a,b)\,x + e_0(a,b),$$

and $e_0(a,b)$, $e_1(a,b)$, and $e_2(a,b)$ are the following polynomials in a and b:

$$\begin{aligned}
e_0(a,b) &= 45\,a^4 + 225\,a^3\,b + 375\,a^2\,b^2 + 225\,a\,b^3 + 45\,b^4,\\
e_1(a,b) &= 39\,a^4 + 367\,a^3\,b + 808\,a^2\,b^2 + 367\,a\,b^3 + 39\,b^4,\\
e_2(a,b) &= 4\,a^4 + 132\,a^3\,b + 448\,a^2\,b^2 + 132\,a\,b^3 + 4\,b^4.
\end{aligned}$$

Remark 14. As mentioned above, if $\sigma_t' = 0$, i.e. in the no-leverage case, the limiting value of the *tricity* is zero. This is a case of *weak* endogeneity analogous to that obtained in Bandi et al., 2023, through a different economic mechanism, namely through *inaction* of the trading process. The case in which $\sigma_t' \neq 0$ gives, instead, *strong* endogeneity.

Remark 15. Stable convergence is derived under the simplifying assumptions in Eq. (22). These assumptions are introduced only to streamline the look of the limiting distribution and should be viewed as being without loss of generality. They solely remove deep leverage effects from the processes σ' and ν, which are now written as being driven by a constant drift and a constant diffusion associated with the orthogonal (between each other and all other processes) Brownian motions Z and B, respectively.

Remark 16. The stable distribution depends on local martingales with respect to both W and V. It also depends on a Brownian motion independent of both W and V. This result, which rests on the interaction between spot volatility and the sampling method, is atypical in high-frequency financial econometrics. Typical stable distributions, in fact, generally depend on W' and, sometimes, on W. The result however exemplifies how consequential endogeneity may be.

Remark 17. We note, however, that if $\sigma' = 0$, i.e., if leverage effects are absent, the stable distribution is only driven by W and W'. This is, in essence, the exogenous case in Li et al. (2025). The intuition for this result works as follows. If $\sigma' = 0$, the sampling times only depend on V, which is independent of W. Therefore, by the joint distributional properties of W and V, i.e., $(W,V) \overset{d}{=} (-W,V)$, we have

$$\left(\Delta_j^n W, \Delta_j^n V\right) \overset{d}{=} \left(-\Delta_j^n W, \Delta_j^n V\right).$$

Now, using the fact that $x \to x^3$ is odd, we also have

$$\mathbb{E}_{j-1}\left[\left(\Delta_j^n W\right)^3\,\Delta_j^n V\right] = \mathbb{E}_{j-1}\left[\left(-\Delta_j^n W\right)^3\,\Delta_j^n V\right] = -\mathbb{E}_{j-1}\left[\left(\Delta_j^n W\right)^3\,\Delta_j^n V\right] = 0.$$

Thus, there cannot be a contribution of V to the distribution of the *tricity* unless $\sigma' \neq 0$. In other words, the process K_t in $\int_0^t K_s\,dV_s$ must be zero if $\sigma' = 0$. It is easy to show that, even in the case $\sigma' \neq 0$, realized variance would be somewhat special in that the orthogonality between W and V would be sufficient to yield an asymptotic stable distribution that only depends on W and W'.

3.5 A taxonomy

Table 2[10] provides a summary of the results. The table relates the form of endogeneity (i.e., the form of hitting time sampling) to (1) the type of endogeneity (weak or strong), (2) the durations drivers (i.e., the relevant Brownian shocks) and (3) implications for the *tricity*.

Sampling rule	Duration driver	Endogeneity type	Implication for tricity
Hitting times for W	Layer-0 shock (W)	Strong	Nonzero centering under asymmetric barriers; inflated/nonstandard variance.
Hitting times for X	Layer-0 and layer-1 shocks (W and V) with V entering at lower order	Strong	Nonzero centering under asymmetric barriers; inflated/nonstandard variance.
Hitting times for $\int \sigma_s ds$	Layer-0 and layer-1 shocks (W and V), both entering at lower order	Weak	Zero centering; inflated/nonstandard variance.
Hitting times for σ	Layer-0 and layer-1 shocks (W and V), both entering at higher order Layer-2 shocks (B and Z), both entering at lower order	Weak or strong	Strong endogeneity, when the layer-1 leverage channel is active; Otherwise, weak.

Table 2: Sampling rules, duration drivers, nature of the endogeneity, and implications for the *tricity*.

4 An economic model

This section applies our methods to the uncertainty zone model of Robert and Rosenbaum (2011). The model provides a revealing bridge between continuous-time arbitrage theory, in which efficient prices are modeled as semi-martingales, and high-frequency financial econometrics, in which transaction prices are noise-contaminated discretely-sampled observations.

We return to the traditional object of interest in the literature, namely realized variance. We do so to better relate some of the implications of our methods to the extant literature: as mentioned, existing work on endogenous sampling focuses on second moments. We also do it because the observation that endogenous sampling does not affect the limiting properties of realized variance rests on observability of the efficient price process. Naturally, lack of observability, a feature of the model in this section, may lead to economically- and statistically- interesting deviations from the typical limit (i.e., integrated variance).

In the model, the price process Y_t is latent. Observed transaction prices P_t are constrained to lie on a discrete grid with tick size $\alpha > 0$:

$$\mathcal{G}_\alpha = \{k\alpha | k \in \mathbb{Z}\}.$$

Unlike in a typical rounding model, in which P_t rounds Y_t to the closest value on the grid, the model hinges on "uncertainty zones" around each mid-tick value $(k + 1/2)\alpha$ defined as:

$$\mathcal{U}_k = ((k + 1/2 - \eta)\alpha, (k + 1/2 + \eta)\alpha),$$

where $\eta \in (0, 1)$.[11] Intuitively, P_t moves from $k\alpha$ to $(k + 1)\alpha$ at the stopping time

$$\tau_{j+1} = \inf\left\{t > \tau_j : Y_t = (k + 1/2 + \eta)\alpha\right\},$$

which means that efficient prices are rounded up to the higher grid when they reach the higher mid-tick value plus

[10] We thank Marcel Rindisbacher for suggesting it.
[11] If $\eta < 1/2$, there is no overlap between the uncertainty zones.

$\eta\alpha$. Because the trigger is not reaching the mid-tick value but an even higher value captured by η, the parameter η represents reluctance to price changes. Similarly, P_t moves from $k\alpha$ to $(k-1)\alpha$ at the stopping time

$$\tau_{j+1} = \inf\left\{t > \tau_j \,:\, Y_t = (k - 1/2 - \eta)\alpha\right\},$$

which means that the efficient prices are rounded up to the lower grid when they reach the lower mid-tick value minus $\eta\alpha$.

More generally, we allow - as in Robert and Rosenbaum (2011) - for discontinuities in the observed prices that are larger than α. As in Jacod (1996), Delattre and Jacod (1997), Rosenbaum (2009), Fukasawa (2010a) and Li and Mykland (2015), *inter alia*, we work with a different α for each $n \in \mathbb{N}$, write α_n and assume $\alpha_n \to 0$ as $n \to \infty$. To this end, define

$$Y^{(\alpha)}_{t,n} \doteq \alpha_n \,\mathrm{Round}\left(\frac{Y_t}{\alpha_n}\right),$$

the rounded efficient price. We assume a sequence of stopping times $\{\tau_{j,n}\}$ recursively defined as

$$\begin{cases} \tau_{0,n} & = 0, \\ \tau_{j+1,n} & = \inf\left\{t > \tau_{j,n} \,:\, Y_t \notin \left(Y^{(\alpha)}_{\tau_{j,n}} - h_{j,n}, Y^{(\alpha)}_{\tau_{j,n}} + h_{j,n}\right)\right\}, \quad j \geq 0, \end{cases} \tag{28}$$

where $h_{j,n}$ is

$$h_{j,n} \doteq \alpha_n \left(L_j - \frac{1}{2} + \eta\right),$$

and $L_j \in \{1, \dots, m\}$ is a random variable known at time $\tau_{j,n}$. At every $\tau_{j,n}$, the value L_j now gives the number of ticks that the latent efficient price process will jump when an update in transaction prices occurs. The update is a function of the efficient price being sufficiently close (i.e., past the closest mid-point on the grid) either to $Y^{(\alpha)}_{\tau_{j,n}} + \alpha_n L_j$ from below or to $Y^{(\alpha)}_{\tau_{j,n}} - \alpha_n L_j$ from above. Again, closeness is represented by η.

As discussed in Robert and Rosenbaum (2011), the model is designed to capture stylized properties of recorded high-frequencies financial prices, in particular rounding on a grid, price discontinuities, bid-ask bounce effects, endogeneity between the price arrivals and the efficient price process as well as durations inversely correlated with the efficient price volatility.

We note that the durations in this model are not defined relative to logarithmic prices. Rather, they are defined with respect to actual prices. We therefore assume the same process X as in Assumption 1 but work with $Y_t \doteq \exp(X_t)$. Itô's formula gives the price-level dynamics as

$$\begin{aligned} dY_t &= \mu^{(Y)}_t \, dt + \sigma^{(Y)}_t \, dW_t, \\ \mu^{(Y)}_t = Y_t \left(\mu_t + \frac{1}{2}\sigma^2_t\right), &\qquad \sigma^{(Y)}_t = Y_t \sigma_t. \end{aligned} \tag{29}$$

Applying the product rule to $\sigma^{(Y)} = Y\sigma$, we obtain

$$d\sigma^{(Y)}_t = \mu'^{(Y)}_t \, dt + \sigma'^{(Y)}_t \, dW_t + \nu^{(Y)}_t \, dV_t,$$

where

$$\begin{aligned} \mu'^{(Y)}_t &= \sigma_t \mu^{(Y)}_t + Y_t \mu'_t + Y_t \sigma_t \sigma'_t, \\ \sigma'^{(Y)}_t &= Y_t(\sigma^2_t + \sigma'_t), \\ \nu^{(Y)}_t &= Y_t \nu_t. \end{aligned}$$

We first derive a *chaotic representation* for the durations $\Delta_{j,n} = \tau_{j,n} - \tau_{j-1,n}$. For convenience, write $Y_j \doteq Y_{\tau_{j,n}}$, $P_j \doteq Y^{(\alpha_n)}_{\tau_{j,n}}$, $h_j \doteq h_{j,n}$ and $z_j \doteq Y_j - P_j$. For $j \geq 1$ and $t \in [\tau_{j-1,n}, \tau_{j,n}]$, define

$$Q_{j-1}(t) \doteq Y_t - P_{j-1}, \qquad G_{j-1}(t) \doteq h^2_{j-1} - Q_{j-1}(t)^2.$$

Lemma 5. *For $j \geq 2$, the stopping times in Eq. (28) satisfy*

$$\tau_{j,n} - \tau_{j-1,n} = \frac{\alpha^2_n L_{j-1}\left(L_{j-1} - 1 + 2\eta\right)}{Y^2_{\tau_{j-1,n}} \sigma^2_{\tau_{j-1,n}}} + \int_{\tau_{j-1,n}}^{\tau_{j,n}} \Phi^{(Y;W)}_{j-1}(t)\, dW_t + \int_{\tau_{j-1,n}}^{\tau_{j,n}} \Phi^{(Y;V)}_{j-1}(t)\, dV_t + \int_{\tau_{j-1,n}}^{\tau_{j,n}} \Phi^{(Y;\Delta)}_{j-1}(t)\, dt \tag{30}$$

where

$$\begin{aligned}
\Phi_{j-1}^{(Y;W)}(t) &= -\frac{2Q_{j-1}(t)}{\sigma_t^{(Y)}} - \frac{2G_{j-1}(t)\sigma_t'^{(Y)}}{\left(\sigma_t^{(Y)}\right)^3} = -\frac{2Q_{j-1}(t)}{Y_t\sigma_t} - \frac{2G_{j-1}(t)(\sigma_t^2+\sigma_t')}{Y_t^2\sigma_t^3}, \\
\Phi_{j-1}^{(Y;V)}(t) &= -\frac{2G_{j-1}(t)\nu_t^{(Y)}}{\left(\sigma_t^{(Y)}\right)^3} = -\frac{2G_{j-1}(t)\nu_t}{Y_t^2\sigma_t^3}, \\
\Phi_{j-1}^{(Y;\Delta)}(t) &= -\frac{2Q_{j-1}(t)\mu_t^{(Y)}}{\left(\sigma_t^{(Y)}\right)^2} - \frac{2G_{j-1}(t)\mu_t'^{(Y)}}{\left(\sigma_t^{(Y)}\right)^3} + \frac{3G_{j-1}(t)\left((\sigma_t'^{(Y)})^2+(\nu_t^{(Y)})^2\right)}{\left(\sigma_t^{(Y)}\right)^4} + \frac{4Q_{j-1}(t)\sigma_t'^{(Y)}}{\left(\sigma_t^{(Y)}\right)^2} \\
&= \frac{Q_{j-1}(t)}{Y_t\sigma_t^2}\left(3\sigma_t^2+4\sigma_t'-2\mu_t\right) + \frac{G_{j-1}(t)}{Y_t^2\sigma_t^4}\left(2\sigma_t^4+4\sigma_t^2\sigma_t'+3((\sigma_t')^2+\nu_t^2)\right) \\
&\quad -\frac{2G_{j-1}(t)}{Y_t^2\sigma_t^4}\left(\sigma_t\mu_t'+\sigma_t^2\mu_t\right). \qquad (31)
\end{aligned}$$

For each fixed j, uniformly in $t \in [\tau_{j-1,n}, \tau_{j,n}]$, we have

$$\Phi_{j-1}^{(Y;W)}(t) = O_p(\alpha_n), \qquad \Phi_{j-1}^{(Y;V)}(t) = O_p(\alpha_n^2), \qquad \Phi_{j-1}^{(Y;\Delta)}(t) = O_p(\alpha_n). \qquad (32)$$

Remark 18. The *chaotic representation* of the durations in Eq. (30) is analogous to that in Eq. (16) for the case of barriers hit by the process X. The definition of the hitting times in this specific model (c.f. Eq. (28)) clarifies how α_n plays the same role as $\sqrt{1/n}$ in Eq. (15). Here, the expectation of the durations between price updates vanishes at rate α_n^2. For the case of barriers hit by the process X, the expectation of the durations vanishes at rate $1/n$.

Remark 19. Similarly, the integrals with respect to W, V and t in Eq. (30) are of order α_n^2, α_n^3 and α_n^3, respectively. As in the case of Eq. (16), because W is a layer-o shock and V is a layer-1 shock, the integral with respect V is lower order relative to the integral with respect to W (c.f. discussion in Remark 7).

Next, we impose more structure on the random tick jump sizes L_j. Assumption 8 is essentially that implied by Eq. (2.4) in Robert and Rosenbaum (2011). It specifies the probabilities of alternative jump sizes as being measurable functions of variables capturing the nature of the trading process.

Assumption 8. *There exists a $\mathcal{F}_t$-adapted continuous process $\{\chi_t\}$ and, for each n, a filtration $\{\mathcal{H}_{j,n}\}_{j\geq 0}$ such that*

1. $\sigma(P_j, L_0, \cdots, L_{j-1}) \subseteq \mathcal{H}_{j,n} \subseteq \mathcal{F}_{\tau_{j,n}}$.
2. *$\chi_{\tau_{j,n}}$ is $\mathcal{H}_{j,n}$-measurable.*
3. *For all $k = 1, \cdots, m$ we have*

$$\mathbb{P}\left[L_j = k \,\middle|\, \mathcal{H}_{j,n}\right] = p_k\left(\chi_{\tau_{j,n}}\right),$$

where $x \to p_k(x)$ are continuous functions such that $p_k(x) \in [0,1]$ for all x and $\sum_{k=1}^m p_k(x) = 1$.

We are now ready to discuss the probability limit of realized variance (whose proof, in the Appendix, is facilitated by the *chaotic representation* of the durations in Lemma 5). Differently from Robert and Rosenbaum (2011), who first estimate the parameter η - in order to recover efficient prices - before computing realized variance, we compute realized variance using observed rounded prices on the grid. In other words, we study the traditional realized variance estimator and simply modify the data generating process.

Theorem 4.1. *Let Assumption 1 hold and let the stopping times be defined as in Eq. (28). Suppose that the transaction at time $t_{j,n}$ reports the price $P_j \doteq Y_{\tau_{j,n}}^{(\alpha_n)}$, where $t_{0,n} = 0$ and $\tau_{j,n} \leq t_{j,n} \leq \tau_{j+1,n}$. Then, for every $t > 0$, the realized variance of the observed logarithmic prices satisfies*

$$\sum_{\substack{j\geq 1 \\ t_{j,n}\leq t}} \left(\log\left(P_j\right) - \log\left(P_{j-1}\right)\right)^2 = \int_0^t \sigma_s^2\, ds + (1-2\eta)\,\alpha_n^2 \sum_{\substack{j\geq 1 \\ t_{j,n}\leq t}} \frac{L_{j-1}}{P_{j-1}^2} + O_p\left(\alpha_n\right). \qquad (33)$$

Under Assumption 8, as $\alpha_n \to 0$, we also have that

$$(1-2\eta)\,\alpha_n^2 \sum_{\substack{j\geq 1\\ t_{j,n}\leq t}} \frac{L_{j-1}}{P_{j-1}^2} \xrightarrow{p} \int_0^t \frac{m_2(\chi_s) - m_\Delta(\chi_s)}{m_\Delta(\chi_s)}\,\sigma_s^2\,ds, \tag{34}$$

where the functions $m_2(x)$ and $m_\Delta(x)$ are defined as follows

$$m_2(x) \doteq \sum_{k=1}^{m} k^2\,p_k(x), \quad m_\Delta(x) \doteq \sum_{k=1}^{m} k\,(k-1+2\eta)\,p_k(x). \tag{35}$$

In particular, combining Eq. (33) with Eq. (34), we obtain

$$\sum_{\substack{j\geq 1\\ t_{j,n}\leq t}} \left(\log\left(P_j\right) - \log\left(P_{j-1}\right)\right)^2 \xrightarrow{p} \int_0^t \frac{m_2(\chi_s)}{m_\Delta(\chi_s)}\,\sigma_s^2\,ds. \tag{36}$$

Eq. (33) documents that rounding introduces a bias which is essentially the sum of the squared deviations between rounded prices on the grid and latent efficient prices. Importantly, if $\eta \neq 1/2$, the bias is a *limiting* bias (i.e., it survives a vanishing tick size $\alpha_n \to 0$) since the left-hand side of Eq. (34) does not vanish with α_n. Relatedly, the sign of the asymptotic contamination in Eq. (36) , i.e., whether integrated variance is upward biased or downward biased, depends on the sign of the parameter η. If η is strictly larger than 1/2, realized variance is downward biased. If η is strictly lower than 1/2, realized variance is upward biased.

The next corollary contains a special case in which the probabilities are not measurable function of an adapted process χ_t. The corollary illustrates the role of η even more explicitly.

Corollary 4.2. *Under the hypotheses of Theorem 4.1, suppose that Assumption 8 holds with $p_k(x) = \pi_k$ for all x and $k = 1, \cdots, m$, where $\pi_k \in [0,1]$ and $\sum_{k=1}^m \pi_k = 1$. Then, for every $t > 0$, as $\alpha_n \to 0$, realized variance converges in probability to*

$$\sum_{\substack{j\geq 1\\ t_{j,n}\leq t}} \left(\log\left(P_j\right) - \log\left(P_{j-1}\right)\right)^2 \xrightarrow{p} \frac{\sum_{k=1}^m k^2\,\pi_k}{\sum_{k=1}^m k\,(k-1+2\eta)\,\pi_k} \int_0^t \sigma_s^2\,ds. \tag{37}$$

In particular, if $L_j = 1$ for every $j \geq 0$, we have

$$\sum_{\substack{j\geq 1\\ t_{j,n}\leq t}} \left(\log\left(P_j\right) - \log\left(P_{j-1}\right)\right)^2 \xrightarrow{p} \frac{1}{2\eta} \int_0^t \sigma_s^2\,ds. \tag{38}$$

Remark 20. The case $\eta = 1/2$, which yields consistency of the realized variance estimator, is related to the approaches in Jacod (1996), Delattre and Jacod (1997), and Li and Mykland (2015). However, differently from these earlier approaches to realized variance estimation with round-off errors, sampling is not deterministic and, therefore, exogenous. As a consequence, the relation between the effective sample size and the size of the discretization (α_n) does not need to be specified in order to derive probability limits.[12] Such a relation is endogenous to the model, as shown explicitly by the *chaotic representation* in Lemma 5, since $\tau_{j,n} - \tau_{j-1,n} = O_p(\alpha_n^2)\ \ \forall j$.

Remark 21. We note that the pure rounding case (i.e., the case $L_j = 1$ with $\eta = 0$) would make realized variance diverge to infinity, thereby completely masking true integrated variance (c.f. Eq. (38)). The reason for this result hinges on the path-wise properties of the underlying (diffusive) efficient price process. When the price process reaches the middle of a tick, it would cross that value an infinite number of times over any fixed period, thereby causing infinite price updates and infinite oscillations between the upper and lower value of the corresponding tick. When $L_j > 1$ with $\eta = 0$, the same degeneracy does not occur, but realized variance is asymptotically upward biased for integrated variance (c.f. Eq. (37)).

[12] Under deterministic sampling, the effective sample size is naturally expressed as being equal to n. The typical specification is $\beta_n = \alpha_n\sqrt{n} \to \beta \in [0,\infty)$, c.f., Delattre and Jacod (1997).

5 Conclusions

We develop a framework for studying high-frequency realized measures (beyond realized variance) under economically-motivated forms of endogenous sampling. Our approach combines the transparency of triangular-array arguments for stable convergence (Jacod, 2008) with the information structure of the Brownian filtration used in the martingale approach to endogenous sampling (e.g., Li et al., 2013). Focusing on the *tricity*, as a maintained example up to Section 4, we illustrate how the method easily delivers possibly-complex stable limit laws. The approach also provides a natural distinction between *weak* and *strong* forms of endogeneity.

A key ingredient is the use of Bartlett-type identities for martingales (Mykland, 1994). We show that these identities can be derived, to any arbitrary order, from the properties of polynomial (Hermite) martingales and establish their connection with *chaotic representations* of powers of Brownian motion. Importantly, the identities provide a convenient way of obtaining the joint moments between price increments and endogenous durations that are needed for asymptotic analysis.

A second key ingredient is *chaotic representations* of the endogenous durations. For the traditional - as well as the novel - forms of hitting-time sampling considered in this article, the representations provide a transparent characterization of how the sampling mechanism interacts with the information in the underlying Brownian filtration. This said, our methods may be extended beyond the particular sampling schemes considered here. Because the representations explicitly characterize the stochastic structure of the durations, they may also be imposed (as we do in this article) as (economically-motivated) *structural* assumptions in place of higher-level conditions on the convergence of moments. To this end, Section 4 applies our methods to a rich data generating process featuring latency of the efficient prices and rounding of the observed prices on a grid, among many other effects.

The analysis is conducted in the continuous setting, in the sense that efficient prices are modeled as being continuous (even in Section 4, where observed prices may vary discontinuously on the assumed grid). Extending the framework to discontinuous efficient price processes represents the natural next step and is left for future work.

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A Proofs

Notation. We work with processes defined on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ verifying standard assumptions. For any given $n \in \mathbb{N}$, we assume the existence of $\mathcal{F}_t$-stopping times $\{\tau_{j,n}\}$, with $0 = \tau_{0,n} < \tau_{1,n} < \cdots < \tau_{j,n} < \cdots$ associated with sampling. For a generic process U, and all $j \geq 1$, $\Delta_j^n U \doteq U_{\tau_{j,n}} - U_{\tau_{j-1,n}}$ denotes increment over the interval $[\tau_{j-1,n}, \tau_{j,n}]$. Consistent with Section 3, $\delta_j^n U(t) \doteq U_t - U_{\tau_{j-1,n}}$, for $t \geq \tau_{j-1,n}$, denotes the corresponding partial increment, so that $\delta_j^n U(\tau_{j,n}) = \Delta_j^n U$. In addition, we define by $\Delta_{j,n} \doteq \tau_{j,n} - \tau_{j-1,n}$ the corresponding duration between arrivals. Finally, for any $t > 0$, we denote by $N_{t,n} \doteq \max\{j \geq 0 | \tau_{j,n} \leq t\} = \sum_{j\geq 1} \mathbb{1}_{\{\tau_{j,n}\leq t\}}$ the process counting the number of sampling times within the interval $[0, t]$. We note that, for any process U, it holds that

$$\sum_{\substack{j\geq 1 \\ \tau_{j,n}\leq t}} U_{\tau_{j,n}} = \int_0^t U_s \, dN_{s,n}.$$

Below, we will sometime use Lemma 4.2 of Robert and Rosenbaum (2012), which we recall here for convenience. The lemma extends the classical Jacod's approach to convergence in probability for triangular arrays to endogenous times.

> **Lemma 4.2 in Robert and Rosenbaum (2012).** *Let $(\mathcal{G}_t)_{t\geq 0}$ be a filtration, $(\nu_{i,n})$ be an array of increasing sequences of stopping times, and $(\xi_{\nu_{i,n}})$ an array of $\mathcal{G}_{\nu_{i,n}}$-measurable random variables and U a random variable. Let ν be another stopping time. Let $N_{n,t} = \sup\{i : \nu_{i,n} \leq t\}$. Assume that*
>
> $$\sum_{\nu_{i-1,n}\leq \nu} \mathbb{E}_{\mathcal{G}_{\nu_{i-1,n}}}\left[\xi^2_{\nu_{i,n}}\right] \xrightarrow{\mathbb{P}} 0,$$
>
> *as $n \to \infty$. Then,*
>
> $$\sum_{\nu_{i,n}\leq \nu} \xi_{\nu_{i,n}} \xrightarrow{\mathbb{P}} U \quad \Longleftrightarrow \quad \sum_{\nu_{i,n}\leq \nu} \mathbb{E}_{\mathcal{G}_{\nu_{i-1,n}}}\left[\xi_{\nu_{i,n}}\right] \xrightarrow{\mathbb{P}} U.$$

Proof of Theorem 2.1. Because $\mathbb{E}[\Delta_{j+1} H_n([M], M) \mid \mathcal{F}_j] = 0$, we only need to show that

$$\mathbb{E}\left[\Delta_{j+1} H_n\left([M], M\right) \Big| \mathcal{F}_j\right] = \mathbb{E}\left[H_n\left(\Delta_{j+1}[M], \Delta_{j+1} M\right) \Big| \mathcal{F}_j\right].$$

To this end, write the following expansion:

$$\exp\left(tx - \frac{1}{2}t^2 a\right) = \sum_{n=0}^{\infty} \frac{t^n}{n!} H_n(a, x). \tag{39}$$

Simple manipulations of Eq. (39) lead to

$$H_n(a_0 + a_1, x_0 + x_1) = \sum_{k=0}^{n} \binom{n}{k} H_{n-k}\left(a_0, x_0\right) H_k\left(a_1, x_1\right).$$

Now, write $a_0 = [M]_j$, $x_0 = M_j$, $a_1 = \Delta_{j+1}[M]$ and $x_1 = \Delta_{j+1} M$.[13] We have

$$H_n\left([M]_{j+1}, M_{j+1}\right) = \sum_{k=0}^{n} \binom{n}{k} H_{n-k}\left([M]_j, M_j\right) H_k\left(\Delta_{j+1}[M], \Delta_{j+1} M\right)$$

and, taking the element $k = 0$ to the left-hand side,

$$\Delta_{j+1} H_n\left([M], M\right) = \sum_{k=1}^{n} \binom{n}{k} H_{n-k}\left([M]_j, M_j\right) H_k\left(\Delta_{j+1}[M], \Delta_{j+1} M\right). \tag{40}$$

For clarity, let us verify the cases $n = 1$ and $n = 2$. If $n = 1$, we have

$$\underbrace{\Delta_{j+1} H_1\left([M], M\right)}_{\Delta_{j+1} M} = \underbrace{H_0\left([M]_j, M_j\right)}_{=1} \underbrace{H_1\left(\Delta_{j+1}[M], \Delta_{j+1} M\right)}_{\Delta_{j+1} M}.$$

[13] The expression in Eq. (39) evaluated at $a_0 = [M]_j$ and $x_0 = M_j$ is the well-known chaotic expansion of the exponential martingale $\exp\left(tM_j - \frac{1}{2}t^2 [M]_j\right)$.

Thus, when $n = 1$, Eq. (40) is satisfied and, importantly,

$$\mathbb{E}\left[H_1\left(\Delta_{j+1}[M], \Delta_{j+1}M\right) \middle| \mathcal{F}_j\right] = 0$$

since

$$\mathbb{E}\left[\Delta_{j+1}H_1\left([M], M\right) \middle| \mathcal{F}_j\right] = 0.$$

The latter result is true - trivially - since $\mathbb{E}[\Delta_{j+1}M \mid \mathcal{F}_j] = 0$ by the martingale property of M. Consider, now, the case $n = 2$. We obtain

$$\begin{aligned}
\underbrace{\Delta_{j+1}H_2\left([M], M\right)}_{M_{j+1}^2 - M_j^2 - ([M]_{j+1} - [M]_j)} &= \sum_{k=1}^{2}\binom{2}{k} H_{2-k}\left([M]_j, M_j\right) H_k\left(\Delta_{j+1}[M], \Delta_{j+1}M\right) \\
&= 2\underbrace{H_1\left([M]_j, M_j\right)}_{M_j}\underbrace{H_1\left(\Delta_{j+1}[M], \Delta_{j+1}M\right)}_{\Delta_{j+1}M} \\
&\quad + \underbrace{H_0\left([M]_j, M_j\right)}_{1}\underbrace{H_2\left(\Delta_{j+1}[M], \Delta_{j+1}M\right)}_{(\Delta_{j+1}M)^2 - \Delta_{j+1}[M]}.
\end{aligned}$$

We now have that the right-hand side coincides with the left-hand side since it can be written as follows:

$$\begin{aligned}
& 2M_j\Delta_{j+1}M + (\Delta_{j+1}M)^2 - \Delta_{j+1}[M] \\
= \;& 2M_jM_{j+1} - 2M_j^2 + M_{j+1}^2 + M_j^2 - 2M_jM_{j+1} - \Delta_{j+1}[M] \\
= \;& M_{j+1}^2 - M_j^2 - ([M]_{j+1} - [M]_j).
\end{aligned}$$

Inverting terms, we notice that

$$\underbrace{H_2\left(\Delta_{j+1}[M], \Delta_{j+1}M\right)}_{(\Delta_{j+1}M)^2 - \Delta_{j+1}[M]} = \underbrace{\Delta_{j+1}H_2\left([M], M\right)}_{M_{j+1}^2 - M_j^2 - ([M]_{j+1} - [M]_j)} - 2\underbrace{H_1\left([M]_j, M_j\right)}_{M_j}\underbrace{H_1\left(\Delta_{j+1}[M], \Delta_{j+1}M\right)}_{\Delta_{j+1}M}. \tag{41}$$

This proves that

$$\mathbb{E}\left[H_2\left(\Delta_{j+1}[M], \Delta_{j+1}M\right) \middle| \mathcal{F}_j\right] = 0$$

since

$$\mathbb{E}\left[\Delta_{j+1}H_2\left([M], M\right) \middle| \mathcal{F}_j\right] = 0$$

and

$$\mathbb{E}\left[H_1\left(\Delta_{j+1}[M], \Delta_{j+1}M\right) \middle| \mathcal{F}_j\right] = 0,$$

which was shown above. In essence, re-writing Eq. (40), we may prove the desired result by iteration after noticing that

$$H_n\left(\Delta_{j+1}[M], \Delta_{j+1}M\right) = \Delta_{j+1}H_n\left([M], M\right) - \sum_{k=1}^{n-1}\binom{n}{k} H_{n-k}\left([M]_j, M_j\right) H_k\left(\Delta_{j+1}[M], \Delta_{j+1}M\right). \tag{42}$$

Hence, for a generic n, we have

$$\mathbb{E}\left[H_n\left(\Delta_{j+1}[M], \Delta_{j+1}M\right) \middle| \mathcal{F}_j\right] = 0$$

since

$$\mathbb{E}\left[\Delta_{j+1}H_n\left([M], M\right) \middle| \mathcal{F}_j\right] = 0$$

and

$$\mathbb{E}\left[H_k\left(\Delta_{j+1}[M], \Delta_{j+1}M\right) \middle| \mathcal{F}_j\right] = 0$$

for $k < n$. The other terms are known given $\mathcal{F}_j$. □

Remark 22. In this remark, we provide an alternative method of proof for the specific data generating process in

Assumption 1. Let $s > 0$ be given. Consider a time $t > s$. Call $\mathbb{E}_s[\cdot] \doteq \mathbb{E}[\cdot \mid \mathcal{F}_s]$ and define

$$\begin{cases} \overline{X}_t \doteq X_t - X_s - \int_s^t \mu_u \, du, \\ \mathrm{Q}_t \doteq \langle X \rangle_t - \langle X \rangle_s = \int_s^t \sigma_u^2 \, du, \\ M_n(t) \doteq H_n(\mathrm{Q}_t, \overline{X}_t). \end{cases}$$

We have

$$\begin{aligned} & dH_{n+1}(\mathrm{Q}_t, \overline{X}_t) = d\left(\overline{X}_t \, H_n(\mathrm{Q}_t, \overline{X}_t) - n \, \mathrm{Q}_t \, H_{n-1}(\mathrm{Q}_t, \overline{X}_t)\right) \\ = \; & H_n(\mathrm{Q}_t, \overline{X}_t) \, d\overline{X}_t + \overline{X}_t \, dH_n(\mathrm{Q}_t, \overline{X}_t) + d\left\langle H_n(\mathrm{Q}_t, \overline{X}_t), \overline{X}_t \right\rangle - n \, H_{n-1}\left(\mathrm{Q}_t, \overline{X}_t\right) d\mathrm{Q}_t - n \, \mathrm{Q}_t \, dH_{n-1}\left(\mathrm{Q}_t, \overline{X}_t\right), \end{aligned}$$

where we used $d\langle \mathrm{Q}_t, H_{n-1}(\mathrm{Q}_t, \overline{X}_t) \rangle \equiv 0$. Applying the bivariate Itô's formula, we obtain

$$\begin{aligned} dH_n(\mathrm{Q}_t, \overline{X}_t) \; = \; & \frac{\partial}{\partial \mathrm{Q}_t} H_n\left(\mathrm{Q}_t, \overline{X}_t\right) d\mathrm{Q}_t + \frac{\partial}{\partial \overline{X}_t} H_n\left(\mathrm{Q}_t, \overline{X}_t\right) d\overline{X}_t + \frac{1}{2} \frac{\partial^2}{\partial \mathrm{Q}_t^2} H_n\left(\mathrm{Q}_t, \overline{X}_t\right) d\langle \mathrm{Q} \rangle_t + \\ & + \frac{1}{2} \frac{\partial^2}{\partial \overline{X}_t^2} H_n\left(\mathrm{Q}_t, \overline{X}_t\right) d\left\langle \overline{X} \right\rangle_t + \frac{\partial^2}{\partial \mathrm{Q}_t \partial \overline{X}_t} H_n\left(\mathrm{Q}_t, \overline{X}_t\right) d\left\langle \mathrm{Q}, \overline{X} \right\rangle_t. \end{aligned}$$

The expression above is simplified by noticing that

$$\begin{cases} d\mathrm{Q}_t = \sigma_t^2 \, dt, \\ d\overline{X}_t = \sigma_t \, dW_t, \\ \langle \mathrm{Q} \rangle_t \equiv 0, \\ \left\langle \overline{X} \right\rangle_t = \langle X \rangle_t - \langle X \rangle_s = \mathrm{Q}_t \Rightarrow d\left\langle \overline{X} \right\rangle_t = d\mathrm{Q}_t = \sigma_t^2 \, dt, \\ \left\langle \mathrm{Q}, \overline{X} \right\rangle_t \equiv 0. \end{cases}$$

Hence,

$$\begin{aligned} dH_n(\mathrm{Q}_t, \overline{X}_t) \; = \; & \sigma_t^2 \frac{\partial}{\partial \mathrm{Q}_t} H_n\left(\mathrm{Q}_t, \overline{X}_t\right) dt + \sigma_t \frac{\partial}{\partial \overline{X}_t} H_n\left(\mathrm{Q}_t, \overline{X}_t\right) dW_t + \frac{\sigma_t^2}{2} \frac{\partial^2}{\partial \overline{X}_t^2} H_n\left(\mathrm{Q}_t, \overline{X}_t\right) dt = \\ = \; & \sigma_t^2 \left(\frac{\partial}{\partial \mathrm{Q}_t} H_n\left(\mathrm{Q}_t, \overline{X}_t\right) + \frac{1}{2} \frac{\partial^2}{\partial \overline{X}_t^2} H_n\left(\mathrm{Q}_t, \overline{X}_t\right) \right) dt + \sigma_t \frac{\partial}{\partial \overline{X}_t} H_n\left(\mathrm{Q}_t, \overline{X}_t\right) dW_t, \end{aligned}$$

Since $H_n(\mathrm{Q}_t, \overline{X}_t)$ is a martingale, it must be that

$$dH_n(\mathrm{Q}_t, \overline{X}_t) = \sigma_t \frac{\partial}{\partial \overline{X}_t} H_n\left(\mathrm{Q}_t, \overline{X}_t\right) dW_t.$$

Similarly,

$$dH_{n-1}(\mathrm{Q}_t, \overline{X}_t) = \sigma_t \frac{\partial}{\partial \overline{X}_t} H_{n-1}\left(\mathrm{Q}_t, \overline{X}_t\right) dW_t.$$

Finally,

$$\begin{aligned} d\left\langle H_n(\mathrm{Q}_t, \overline{X}_t), \overline{X}_t \right\rangle \; = \; & dH_n(\mathrm{Q}_t, \overline{X}_t) \, d\overline{X}_t = \\ = \; & \sigma_t^2 \frac{\partial}{\partial \overline{X}_t} H_n\left(\mathrm{Q}_t, \overline{X}_t\right) (dW_t)^2 = \sigma_t^2 \frac{\partial}{\partial \overline{X}_t} H_n\left(\mathrm{Q}_t, \overline{X}_t\right) dt \end{aligned} \tag{43}$$

In sum,

$$\begin{aligned}
dH_{n+1} &= H_n(\mathrm{Q}_t,\overline{X}_t)\,\sigma_t\,dW_t + \overline{X}_t\,\sigma_t\,\frac{\partial}{\partial\overline{X}_t}H_n\left(\mathrm{Q}_t,\overline{X}_t\right)\,dW_t + \\
&\quad +\sigma_t^2\,\frac{\partial}{\partial\overline{X}_t}H_n\left(\mathrm{Q}_t,\overline{X}_t\right)\,dt - n\,\sigma_t^2\,H_{n-1}\left(\mathrm{Q}_t,\overline{X}_t\right)\,dt - n\,\mathrm{Q}_t\,\sigma_t\,\frac{\partial}{\partial\overline{X}_t}H_{n-1}\left(\mathrm{Q}_t,\overline{X}_t\right)\,dW_t \\
&= \sigma_t^2\left(\frac{\partial}{\partial\overline{X}_t}H_n\left(\mathrm{Q}_t,\overline{X}_t\right) - n\;H_{n-1}\left(\mathrm{Q}_t,\overline{X}_t\right)\right)dt \\
&\quad +\sigma_t\,H_n(\mathrm{Q}_t,\overline{X}_t)\;dW_t + \overline{X}_t\,\sigma_t\,\frac{\partial}{\partial\overline{X}_t}H_n\left(\mathrm{Q}_t,\overline{X}_t\right)\,dW_t - n\,\mathrm{Q}_t\,\sigma_t\,\frac{\partial}{\partial\overline{X}_t}H_{n-1}\left(\mathrm{Q}_t,\overline{X}_t\right)\,dW_t \\
&= \sigma_t\left(H_n(\mathrm{Q}_t,\overline{X}_t) + \overline{X}_t\,\frac{\partial}{\partial\overline{X}_t}H_n\left(\mathrm{Q}_t,\overline{X}_t\right) - n\,\mathrm{Q}_t\,\frac{\partial}{\partial\overline{X}_t}H_{n-1}\left(\mathrm{Q}_t,\overline{X}_t\right)\right)dW_t.
\end{aligned}$$

Thus,

$$\mathbb{E}\left[H_n\left(\mathrm{Q}_t,\overline{X}_t\right)\,\middle|\,\mathcal{F}_s\right] = H_n\left(\mathrm{Q}_s,\overline{X}_s\right), \qquad 0\le s\le t.$$

To proceed with the remaining proofs, we need to state two technical lemmas that will be used on several occasions.

Lemma 6. *For any sampling scheme and any Brownian motion Q, we have*

$$\mathbb{E}_{\tau_{j-1,n}}\left[\int_{\tau_{j-1,n}}^{\tau_{j,n}}\left(Q_s - Q_{\tau_{j-1,n}}\right)\,ds\right] = \mathbb{E}_{\tau_{j-1,n}}\left[\left(\tau_{j,n}-\tau_{j-1,n}\right)\left(Q_{\tau_{j,n}} - Q_{\tau_{j-1,n}}\right)\right]. \tag{44}$$

Also, if $\{\eta\}$ is an adapted bounded càdlàg process, we have

$$\mathbb{E}_{\tau_{j-1,n}}\left[\left(\int_{\tau_{j-1,n}}^{\tau_{j,n}}\eta_s\,dQ_s\right)^3\right] = 3\,\mathbb{E}_{\tau_{j-1,n}}\left[\int_{\tau_{j-1,n}}^{\tau_{j,n}}\eta_s^2\left(\int_{\tau_{j-1,n}}^{s}\eta_u\,dQ_u\right)ds\right]. \tag{45}$$

Proof. Let $u(s)=s$. Since $[Q,u]_s=0$, we have that $Q_s\,ds = d(s\,Q_s) - s\,dQ_s$. Thus,

$$\begin{aligned}
&\mathbb{E}_{\tau_{j-1,n}}\left[\int_{\tau_{j-1,n}}^{\tau_{j,n}}(Q_s - Q_{\tau_{j-1,n}})\,ds\right] \\
&= \mathbb{E}_{\tau_{j-1,n}}\left[\int_{\tau_{j-1,n}}^{\tau_{j,n}}Q_s\,ds - Q_{\tau_{j-1,n}}\,\Delta_{j,n}\right] \\
&= \mathbb{E}_{\tau_{j-1,n}}\left[\tau_{j,n}\,Q_{\tau_{j,n}} - \tau_{j-1,n}\,Q_{\tau_{j-1,n}} - \int_{\tau_{j-1,n}}^{\tau_{j,n}}s\,dQ_s - Q_{\tau_{j-1,n}}\,\Delta_{j,n}\right] \\
&= \mathbb{E}_{\tau_{j-1,n}}\left[\tau_{j,n}\,Q_{\tau_{j,n}} - \tau_{j,n}\,Q_{\tau_{j-1,n}} + \tau_{j,n}\,Q_{\tau_{j-1,n}} - \tau_{j-1,n}\,Q_{\tau_{j-1,n}} - \int_{\tau_{j-1,n}}^{\tau_{j,n}}s\,dQ_s - Q_{\tau_{j-1,n}}\,\Delta_{j,n}\right] \\
&= \mathbb{E}_{\tau_{j-1,n}}\left[\tau_{j,n}\left(Q_{\tau_{j,n}} - Q_{\tau_{j-1,n}}\right) + Q_{\tau_{j-1,n}}\,\Delta_{j,n} - \int_{\tau_{j-1,n}}^{\tau_{j,n}}s\,dQ_s - Q_{\tau_{j-1,n}}\,\Delta_{j,n}\right] \\
&= \mathbb{E}_{\tau_{j-1,n}}\left[\tau_{j,n}\left(Q_{\tau_{j,n}} - Q_{\tau_{j-1,n}}\right)\right] \\
&= \mathbb{E}_{\tau_{j-1,n}}\left[\left(\tau_{j,n}-\tau_{j-1,n}\right)\left(Q_{\tau_{j,n}} - Q_{\tau_{j-1,n}}\right)\right] + \tau_{j-1,n}\,\mathbb{E}_{\tau_{j-1,n}}\left[Q_{\tau_{j,n}} - Q_{\tau_{j-1,n}}\right] \\
&= \mathbb{E}_{\tau_{j-1,n}}\left[\left(\tau_{j,n}-\tau_{j-1,n}\right)\left(Q_{\tau_{j,n}} - Q_{\tau_{j-1,n}}\right)\right],
\end{aligned}$$

which is Eq. (44). Now, notice that

$$\begin{aligned}
\mathbb{E}_{\tau_{j-1,n}}\left[\left(\int_{\tau_{j-1,n}}^{\tau_{j,n}} \eta_s\, dQ_s\right)^3\right] &= 3\,\mathbb{E}_{\tau_{j-1,n}}\left[\int_{\tau_{j-1,n}}^{\tau_{j,n}} \eta_s^2\, ds \int_{\tau_{j-1,n}}^{\tau_{j,n}} \eta_s\, dQ_s\right] \\
&= 3\,\mathbb{E}_{\tau_{j-1,n}}\left[\int_{\tau_{j-1,n}}^{\tau_{j,n}} \eta_s^2 \left(\int_{\tau_{j-1,n}}^{s} \eta_u\, dQ_u\right) ds + \int_{\tau_{j-1,n}}^{\tau_{j,n}} \eta_s \left(\int_{\tau_{j-1,n}}^{s} \eta_u^2\, du\right) dQ_s\right] \\
&= 3\,\mathbb{E}_{\tau_{j-1,n}}\left[\int_{\tau_{j-1,n}}^{\tau_{j,n}} \eta_s^2 \left(\int_{\tau_{j-1,n}}^{s} \eta_u\, dQ_u\right) ds\right],
\end{aligned}$$

where the first equation derives from the third Bartlett identity, the second equation derives from stochastic integration by part with $[\int_0^{\cdot} \eta_s^2\, ds, \int_0^{\cdot} \eta_s\, dQ_s]_t \equiv 0$ and the last equation is a consequence of the first Bartlett identity. Eq. (45) is then proved. □

Lemma 7. *Let $\{\mathcal{F}_t\}$ be the filtration defined in Assumption 2 and let $\{\tau_{j,n}, j \geq 0\}$ be a sequence of observation times that are all $\mathcal{F}_t$-stopping times. Assume that the durations $\Delta_{j,n} = \tau_{j,n} - \tau_{j-1,n}$ can be represented as*

$$\Delta_{j,n} = \frac{h_{\tau_{j-1,n}}}{n} + \int_{\tau_{j-1,n}}^{\tau_{j,n}} \Phi_{j,n}^{(\Delta)}(t)\, dt + \int_{\tau_{j-1,n}}^{\tau_{j,n}} \Phi_{j,n}^{(W)}(t)\, dW_t + \int_{\tau_{j-1,n}}^{\tau_{j,n}} \Phi_{j,n}^{(V)}(t)\, dV_t + \sum_{k=1}^{\mathrm{p}} \int_{\tau_{j-1,n}}^{\tau_{j,n}} \Phi_{j,n}^{\left(W^{(k)}\right)}(t)\, dW_t^{(k)}, \quad (46)$$

where h is a bounded adapted càdlàg process and $\{\Phi_{j,n}^{(Z)}\}$, with Z being any of the Brownian motions that generate $\{\mathcal{F}_t\}$, is a bounded adapted càdlàg process defined on $[\tau_{j-1,n}, \tau_{j,n}]$ such that $\Phi_{j,n}^{(Z)}(t) = O_p(n^{-1/2})$ uniformly on $[\tau_{j-1,n}, \tau_{j,n}]$. Also, $\Phi_{j,n}^{(\Delta)}(t) = O_p(n^{-1/2})$ uniformly on $[\tau_{j-1,n}, \tau_{j,n}]$. Then, for any bounded càdlàg process G, we obtain

$$\frac{1}{n} \sum_{\tau_{j,n} \leq t} G_{\tau_{j-1,n}} \xrightarrow{p} \int_0^t \frac{G_s}{h_s}\, ds. \quad (47)$$

In particular, for $G_t \equiv 1$ for all t, we have

$$\frac{1}{n} \sum_{\tau_{j,n} \leq t} 1 = \frac{N_{t,n}}{n} \xrightarrow{p} \int_0^t \frac{1}{h_s}\, ds.$$

Proof. From Eq. (46), we obtain the decomposition

$$\frac{1}{n} \sum_{\tau_{j,n} \leq t} G_{\tau_{j-1,n}} = \sum_{\tau_{j,n} \leq t} \frac{G_{\tau_{j-1,n}}}{h_{\tau_{j-1,n}}} \frac{h_{\tau_{j-1,n}}}{n} = \sum_{\tau_{j,n} \leq t} \frac{G_{\tau_{j-1,n}}}{h_{\tau_{j-1,n}}} \Delta_{j,n} - \left(F_{n,t} + M_{n,t}^{(W)} + M_{n,t}^{(V)} + \sum_{k=1}^{\mathrm{p}} M_{n,t}^{(W^{(k)})}\right),$$

where

$$\begin{aligned}
F_{n,t} &\doteq \sum_{\tau_{j,n} \leq t} \frac{G_{\tau_{j-1,n}}}{h_{\tau_{j-1,n}}} \int_{\tau_{j-1,n}}^{\tau_{j,n}} \Phi_{j,n}^{(\Delta)}(s)\, ds, \\
M_{n,t}^{(W)} &\doteq \sum_{\tau_{j,n} \leq t} \frac{G_{\tau_{j-1,n}}}{h_{\tau_{j-1,n}}} \int_{\tau_{j-1,n}}^{\tau_{j,n}} \Phi_{j,n}^{(W)}(s)\, dW_s, \\
M_{n,t}^{(V)} &\doteq \sum_{\tau_{j,n} \leq t} \frac{G_{\tau_{j-1,n}}}{h_{\tau_{j-1,n}}} \int_{\tau_{j-1,n}}^{\tau_{j,n}} \Phi_{j,n}^{(V)}(s)\, dV_s,
\end{aligned}$$

and similar definitions hold for all other remaining terms. Consider, first, the term $M_{n,t}^{(W)}$ and notice that

$$\begin{aligned}\mathbb{E}\left[\left(M_{n,t}^{(W)}\right)^2\right] &= \underbrace{\mathbb{E}\left[\sum_{\tau_{j,n}\leq t}\frac{G_{\tau_{j-1,n}}^2}{h_{\tau_{j-1,n}}^2}\left(\int_{\tau_{j-1,n}}^{\tau_{j,n}}\Phi_{j,n}^{(W)}(s)\,dW_s\right)^2\right]}_{A_{n,t}}+\\ &\quad+2\underbrace{\mathbb{E}\left[\sum_{\tau_{j,n}<\tau_{k,n}\leq t}\frac{G_{\tau_{j-1,n}}}{h_{\tau_{j-1,n}}}\frac{G_{\tau_{k-1,n}}}{h_{\tau_{k-1,n}}}\left(\int_{\tau_{j-1,n}}^{\tau_{j,n}}\Phi_{j,n}^{(W)}(s)\,dW_s\right)\left(\int_{\tau_{k-1,n}}^{\tau_{k,n}}\Phi_{k,n}^{(W)}(u)\,dW_u\right)\right]}_{A'_{n,t}}.\end{aligned}$$

The coefficient $A_{n,t}$ verifies

$$\begin{aligned}A_{n,t} &= \sum_{\tau_{j,n}\leq t}\mathbb{E}\left[\frac{G_{\tau_{j-1,n}}^2}{h_{\tau_{j-1,n}}^2}\mathbb{E}_{\tau_{j-1,n}}\left[\left(\int_{\tau_{j-1,n}}^{\tau_{j,n}}\Phi_{j,n}^{(W)}(s)\,dW_s\right)^2\right]\right]=\sum_{\tau_{j,n}\leq t}\mathbb{E}\left[\frac{G_{\tau_{j-1,n}}^2}{h_{\tau_{j-1,n}}^2}\mathbb{E}_{\tau_{j-1,n}}\left[\left(\int_{\tau_{j-1,n}}^{\tau_{j,n}}\left(\Phi_{j,n}^{(W)}(s)\right)^2\,ds\right)\right]\right]\\ &= O_p\left(n^{-1}\right)\xrightarrow{p}0.\end{aligned}\tag{48}$$

Since, by proper conditioning and the tower rule, $A'_{n,t}\equiv 0$, we obtain $M_{n,t}^{(W)}\xrightarrow{p}0$. Similarly, $M_{n,t}^{(V)}\xrightarrow{p}0$ and the same convergence holds for all other remaining terms. Finally, we have $F_{n,t}=O_p(n^{-1/2})\xrightarrow{p}0$. In sum,

$$\frac{1}{n}\sum_{\tau_{j,n}\leq t}G_{\tau_{j-1,n}}=\sum_{\tau_{j,n}\leq t}\frac{G_{\tau_{j-1,n}}}{h_{\tau_{j-1,n}}}\Delta_{j,n}+o_p(1)\xrightarrow{p}\int_0^t\frac{G_s}{h_s}\,ds,$$

as claimed. □

Proof of Lemma 1. In what follows, in order to simplify the notation, we use the convention $U_n\doteq a/\sqrt{n}$ and $\ell_n\doteq b/\sqrt{n}$. Fix $j\geq 1$ and for $t\geq\tau_{j-1,n}$, let $\delta_j^nW(t)\doteq W_t-W_{\tau_{j-1,n}}$. Hence, $\delta_j^nW(\tau_{j-1,n})=0$. Note that $\delta_j^nW(\tau_{j,n})=+U_n$ or $\delta_j^nW(\tau_{j,n})=-\ell_n$ and δ_j^nW is continuous as W. It is not difficult to see that $\delta_j^nW(t)=O_p(n^{-1/2})$ uniformly on $[\tau_{j-1,n},\tau_{j,n}]$[14]. Consider, for all $n\in\mathbb{N}$, the function

$$u^{(n)}(y)\doteq(U_n-y)(\ell_n+y)=\left(U_n\ell_n+(U_n-\ell_n)\,y-y^2\right),$$

which has the following first and second order partial derivatives

$$\begin{cases}u_y^{(n)}=(U_n-\ell_n-2\,y),\\ u_{yy}^{(n)}=-2.\end{cases}\tag{49}$$

In addition, we have the following boundary conditions

$$\begin{cases}u^{(n)}\left(\delta_j^nW\left(\tau_{j,n}\right)\right)=0,\\ u^{(n)}\left(\delta_j^nW\left(\tau_{j-1,n}\right)\right)=u^{(n)}(0)=a\,b/n.\end{cases}\tag{50}$$

Itô's lemma for $Z_t\doteq u^{(n)}(\delta_j^nW(t))$ gives

$$\begin{aligned}dZ_t &= u_y^{(n)}\,d\left(\delta_j^nW(t)\right)+\frac{1}{2}u_{yy}^{(n)}\,d\left[\delta_j^nW\right]_t\\ &= u_y^{(n)}dW_t-dt.\end{aligned}$$

[14]Fix $\omega\in\Omega$. We have that $\delta_j^nW(t)(\omega)\in[-\ell_n,+U_n]$ on $\{\tau_{j-1,n}\leq t\leq\tau_{j,n}\}$. Therefore, path-wise continuity guarantees that $\delta_j^nW(\tau_{j,n})\in\{-\ell_n,+U_n\}$ and

$$\sup_{\tau_{j-1,n}(\omega)\leq t\leq\tau_{j,n}(\omega)}\left|\delta_j^nW(t)(\omega)\right|\leq\frac{2\,(a\vee b)}{\sqrt{n}}\Rightarrow\sqrt{n}\,Z_n(\omega)\leq M,$$

where $Z_n(\omega)\doteq\sup\{|\delta_j^nW(t)(\omega)||\tau_{j-1,n}(\omega)\leq t\leq\tau_{j,n}(\omega)\}$ and $M\doteq 2\,(a\vee b)$.

Integrating from $\tau_{j-1,n}$ to $\tau_{j,n}$, and using the boundary conditions in Eq. (50), we obtain

$$\tau_{j,n} - \tau_{j-1,n} = \frac{ab}{n} + \int_{\tau_{j-1,n}}^{\tau_{j,n}} u_y^{(n)}\left(\delta_j^n W(t)\right) dW_t. \tag{51}$$

Since, as mentioned earlier, $\delta_j^n W(t) = O_p(n^{-1/2})$ uniformly on $[\tau_{j-1,n}, \tau_{j,n}]$, from $u_y^{(n)}(y)$ in Eq. (49), we obtain the following probability order

$$u_y^{(n)}\left(\delta_j^n W(t)\right) = \left(\frac{a-b}{\sqrt{n}} - 2\,\delta_j^n W(t)\right) = O_p\left(n^{-1/2}\right).$$

uniformly on $[\tau_{j-1,n}, \tau_{j,n}]$. □

Proof of Theorem 3.1. Call $N_{t,n} \doteq \inf\{j|\tau_{j,n} > t\} - 1$ the total number of random arrivals within the interval $[0,t]$. It is not difficult to verify that the sampling scheme in Assumption 3 satisfies

$$\frac{N_{t,n}}{n} \xrightarrow{p} \frac{t}{a\,b} \quad \text{as } n \to \infty. \tag{52}$$

Furthermore, we have that

$$W_{\tau_{j,n}} - W_{\tau_{j-1,n}} \overset{d}{=} n^{-1/2} Z_j,$$

where the $\{Z_j\}$s are iid random variables with point mass in a and $-b$ and $\mathbb{P}[Z_j = a] = b/(a+b)$. Accordingly, the following moments apply:

$$\begin{aligned}
\mathbb{E}_{\tau_{j-1,n}}\left[W_{\tau_{j,n}} - W_{\tau_{j-1,n}}\right] &= 0,\\
\mathbb{E}_{\tau_{j-1,n}}\left[\left(W_{\tau_{j,n}} - W_{\tau_{j-1,n}}\right)^2\right] &= n^{-1}\,a\,b,\\
\mathbb{E}_{\tau_{j-1,n}}\left[\left(W_{\tau_{j,n}} - W_{\tau_{j-1,n}}\right)^3\right] &= n^{-3/2}\,a\,b\,(a-b),\\
\mathbb{E}_{\tau_{j-1,n}}\left[\left(W_{\tau_{j,n}} - W_{\tau_{j-1,n}}\right)^4\right] &= n^{-2}\,a\,b\,\left(a^2 - ab + b^2\right),\\
\mathbb{E}_{\tau_{j-1,n}}\left[\left(W_{\tau_{j,n}} - W_{\tau_{j-1,n}}\right)^5\right] &= n^{-5/2}\,a\,b\,(a-b)\left(a^2+b^2\right),\\
\mathbb{E}_{\tau_{j-1,n}}\left[\left(W_{\tau_{j,n}} - W_{\tau_{j-1,n}}\right)^6\right] &= \frac{ab\left(a^5+b^5\right)}{n^3(a+b)}.
\end{aligned} \tag{53}$$

Thus,

$$\begin{aligned}
n^{1/2}\sum_{\tau_{j,n}\le t}\mathbb{E}_{\tau_{j-1,n}}\left[\left(W_{\tau_{j,n}} - W_{\tau_{j-1,n}}\right)^3\right] &= n^{1/2}\sum_{\tau_{j,n}\le t} n^{-3/2}\,a\,b\,(a-b) = \sum_{\tau_{j,n}\le t} a\,b\,(a-b)\,\frac{1}{n}\\
&= \frac{N_{t,n}}{n}\,a\,b\,(a-b) \xrightarrow{p} t\,(a-b),
\end{aligned}$$

where we have used the convergence in Eq. (52). Similarly,

$$\begin{aligned}
n\sum_{\tau_{j,n}\le t}\mathbb{E}_{\tau_{j-1,n}}\left[\left(W_{\tau_{j,n}} - W_{\tau_{j-1,n}}\right)^4\right] &= n\sum_{\tau_{j,n}\le t} n^{-2}\,a\,b\,\left(a^2 - ab + b^2\right) = \sum_{\tau_{j,n}\le t} a\,b\,\left(a^2 - ab + b^2\right)\frac{1}{n}\\
&= \frac{N_{t,n}}{n}\,a\,b\,\left(a^2 - ab + b^2\right) \xrightarrow{p} t\,\left(a^2 - ab + b^2\right).
\end{aligned}$$

Consider, now, the decomposition $\Omega = A \cup A^c$ with $A = \{\Delta_j^n W = n^{-1/2}\,a\}$ and $A^c = \{\Delta_j^n W = -n^{-1/2}\,b\}$. Using $1 = \mathbb{1}_{\{A\}} + \mathbb{1}_{\{A^c\}}$, we obtain

$$\begin{aligned}
\mathbb{E}_{\tau_{j-1,n}}\left[\Delta_{j,n}\right] &= \mathbb{E}_{\tau_{j-1,n}}\left[\Delta_{j,n}; \Delta_j^n W = n^{-1/2}\,a\right] + \mathbb{E}_{\tau_{j-1,n}}\left[\Delta_{j,n}; \Delta_j^n W = -n^{-1/2}\,b\right]\\
&= n^{-1}\,a\,b,\\
\mathbb{E}_{\tau_{j-1,n}}\left[\Delta_{j,n}\,\Delta_j^n W\right] &= n^{-1/2}\,a\,\mathbb{E}_{\tau_{j-1,n}}\left[\Delta_{j,n}; \Delta_j^n W = n^{-1/2}\,a\right] - n^{-1/2}\,b\,\mathbb{E}_{\tau_{j-1,n}}\left[\Delta_{j,n}; \Delta_j^n W = -n^{-1/2}\,b\right]\\
&= \frac{1}{3\,n^{3/2}}\,a\,b\,(a-b),
\end{aligned} \tag{54}$$

given the second and the third Bartlett identities. In order to compute the expected values

$$\mathbb{E}_{\tau_{j-1,n}}\left[\Delta_{j,n}; \Delta_j^n W = n^{-1/2}\, a\right],$$
$$\mathbb{E}_{\tau_{j-1,n}}\left[\Delta_{j,n}; \Delta_j^n W = -n^{-1/2}\, b\right]$$

we solve the following system of equations

$$\begin{cases} n^{-1/2}\,(a\,x - b\,y) = \frac{1}{3\,n^{3/2}}\, a\,b\,(a-b), \\ x + y = n^{-1}\, a\,b, \end{cases}$$

giving

$$\begin{cases} \mathbb{E}_{\tau_{j-1,n}}\left[\Delta_{j,n}; \Delta_j^n W = n^{-1/2}\, a\right] = \frac{a\,b\,(a+2\,b)}{3\,(a+b)\,n} \\ \mathbb{E}_{\tau_{j-1,n}}\left[\Delta_{j,n}; \Delta_j^n W = -n^{-1/2}\, b\right] = \frac{a\,b\,(2\,a+b)}{3\,(a+b)\,n}. \end{cases}$$

As a by-product, we compute the following cross-moments

$$\begin{aligned} \mathbb{E}_{\tau_{j-1,n}}\left[\Delta_{j,n}\left(\Delta_j^n W\right)^2\right] &= n^{-1}\, a^2\, \mathbb{E}_{\tau_{j-1,n}}\left[\Delta_{j,n}; \Delta_j^n W = n^{-1/2}\, a\right] + n^{-1}\, b^2\, \mathbb{E}_{\tau_{j-1,n}}\left[\Delta_{j,n}; \Delta_j^n W = -n^{-1/2}\, b\right] \\ &= \frac{ab\left(a^2 + ab + b^2\right)}{3n^2}, \end{aligned} \tag{55}$$

and

$$\begin{aligned} \mathbb{E}_{\tau_{j-1,n}}\left[\Delta_{j,n}\left(\Delta_j^n W\right)^3\right] &= n^{-3/2}\, a^3\, \mathbb{E}_{\tau_{j-1,n}}\left[\Delta_{j,n}; \Delta_j^n W = n^{-1/2}\, a\right] - n^{-3/2}\, b^3\, \mathbb{E}_{\tau_{j-1,n}}\left[\Delta_{j,n}; \Delta_j^n W = -n^{-1/2}\, b\right] \\ &= \frac{ab(a-b)(a+b)^2}{3n^{5/2}}. \end{aligned} \tag{56}$$

Using the fourth Bartlett identity and Eq. (55), we obtain

$$\mathbb{E}_{\tau_{j-1,n}}\left[\Delta_{j,n}^2\right] = \frac{ab\left(a^2 + 3ab + b^2\right)}{3n^2}, \tag{57}$$

which leads to

$$n \sum_{\tau_{j,n} \le t} \mathbb{E}_{\tau_{j-1,n}}\left[\Delta_{j,n}^2\right] = \frac{N_{t,n}}{n}\, \frac{ab\left(a^2 + 3ab + b^2\right)}{3} \xrightarrow{p} \frac{t\,\left(a^2 + 3ab + b^2\right)}{3}.$$

Using the fifth Bartlett identity, we have

$$\mathbb{E}_{\tau_{j-1,n}}\left[\Delta_{j,n}^2\, \Delta_j^n W\right] = \frac{ab(a-b)\left(7a^2 + 20ab + 7b^2\right)}{45n^{5/2}},$$

which leads to

$$n^{3/2} \sum_{\tau_{j,n} \le t} \mathbb{E}_{\tau_{j-1,n}}\left[\Delta_{j,n}^2\, \Delta_j^n W\right] \xrightarrow{p} t\, \frac{(a-b)\left(7a^2 + 20ab + 7b^2\right)}{45}.$$

To prove stable convergence, we verify the criteria in Theorem 1 of Podolskij and Vetter (2010). Define $\beta_{j,n} = \sigma_{\tau_{j-1,n}} \Delta_j^n W$ and note that

$$\begin{aligned} X_{\tau_{j,n}} - X_{\tau_{j-1,n}} &= \int_{\tau_{j-1,n}}^{\tau_{j,n}} \mu_u\, du + \int_{\tau_{j-1,n}}^{\tau_{j,n}} \sigma_u\, dW_u = \int_{\tau_{j-1,n}}^{\tau_{j,n}} \mu_u\, du + \int_{\tau_{j-1,n}}^{\tau_{j,n}} \left(\sigma_u - \sigma_{\tau_{j-1,n}} + \sigma_{\tau_{j-1,n}}\right) dW_u \\ &= \beta_{j,n} + \int_{\tau_{j-1,n}}^{\tau_{j,n}} \mu_u\, du + \int_{\tau_{j-1,n}}^{\tau_{j,n}} \left(\sigma_s - \sigma_{\tau_{j-1,n}}\right) dW_s = \beta_{j,n} + \zeta_{j,n}(1) + \zeta_{j,n}(2) \doteq \beta_{j,n} + R_{j,n}, \end{aligned} \tag{58}$$

where the triangular arrays $\{\zeta_{j,n}(1)\}$ and $\{\zeta_{j,n}(2)\}$ are

$$\begin{aligned}
\zeta_{j,n}(1) &= \int_{\tau_{j-1,n}}^{\tau_{j,n}} \left(\mu_u - \mu_{\tau_{j-1,n}}\right) du \\
&+ \int_{\tau_{j-1,n}}^{\tau_{j,n}} \left[\int_{\tau_{j-1,n}}^{u} \mu'_s \, ds + \int_{\tau_{j-1,n}}^{u} \left(\sigma'_s - \sigma'_{\tau_{j-1,n}}\right) dW_s + \int_{\tau_{j-1,n}}^{u} \left(\nu_s - \nu_{\tau_{j-1,n}}\right) dV_s \right] dW_u, \\
\zeta_{j,n}(2) &= \int_{\tau_{j-1,n}}^{\tau_{j,n}} \mu_{\tau_{j-1,n}} \, du + \sigma'_{\tau_{j-1,n}} \int_{\tau_{j-1,n}}^{\tau_{j,n}} \left(W_u - W_{\tau_{j-1,n}}\right) dW_u + \nu_{\tau_{j-1,n}} \int_{\tau_{j-1,n}}^{\tau_{j,n}} \left(V_u - V_{\tau_{j-1,n}}\right) dW_u.
\end{aligned}$$

Note that $\zeta_{j,n}(2) = O_p(n^{-1})$ and $\zeta_{j,n}(1) = O_p(n^{-1})$ or $\zeta_{j,n}(1) = o_p(n^{-1})$ depending on the assumptions on the porcesses μ, μ', σ' and ν. We thus have, in both cases, that $R_{j,n} = \zeta_{j,n}(1) + \zeta_{j,n}(2) = O_p(n^{-1})$, giving

$$\left(\Delta_j^n X\right)^3 = \underbrace{\sigma^3_{\tau_{j-1,n}} \left(\Delta_j^n W\right)^3}_{O_p(n^{-3/2})} + \underbrace{R^3_{j,n}}_{O_p(n^{-3})} + 3 \underbrace{\sigma^2_{\tau_{j-1,n}} \left(\Delta_j^n W\right)^2 R_{j,n}}_{O_p(n^{-2})} + 3 \underbrace{\sigma_{\tau_{j-1,n}} \Delta_j^n W \, R^2_{j,n}}_{O_p(n^{-5/2})} = \sigma^3_{\tau_{j-1,n}} \left(\Delta_j^n W\right)^3 + O_p\left(n^{-2}\right), \tag{59}$$

which yields

$$\begin{aligned}
\sum_{\tau_{j,n} \le t} n^{1/2} \, \mathbb{E}_{\tau_{j-1,n}} \left[\left(\Delta_j^n X\right)^3 \right] &= \sum_{\tau_{j,n} \le t} n^{1/2} \sigma^3_{\tau_{j-1,n}} \, \mathbb{E}_{\tau_{j-1,n}} \left[\left(\Delta_j^n W\right)^3 \right] + O_p\left(n^{-1}\right) \\
&= a\, b\, (a-b) \frac{N_{t,n}}{t\, n} \sum_{\tau_{j,n} \le t} \sigma^3_{\tau_{j-1,n}} \frac{t}{N_{t,n}} + O_p\left(n^{-1}\right) \xrightarrow{p} (a-b) \int_0^t \sigma_s^3 \, ds,
\end{aligned}$$

by the convergence in Eq. (52). Using $\Delta_j^n X = \beta_{j,n} + R_{j,n}$, we have the following decomposition

$$\begin{aligned}
& n^{1/2} \left(\sum_{\tau_{j,n} \le t} n^{1/2} \left(\Delta_j^n X\right)^3 - (a-b) \int_0^t \sigma_s^3 \, ds \right) \\
&= \sum_{\tau_{j,n} \le t} \xi_{j,n} + n^{1/2} \, (a-b) \left(\sum_{\tau_{j,n} \le t} \sigma^3_{\tau_{j-1,n}} \Delta_{j,n} - \int_0^t \sigma_s^3 \, ds \right) + O_p\left(n^{-1/2}\right), \\
&= \sum_{\tau_{j,n} \le t} \xi_{j,n} + \sum_{\tau_{j,n} \le t} \xi'_{j,n} - (a-b) \, n^{1/2} \int_{\tau_{N_{t,n},n}}^t \sigma_s^3 \, ds + O_p\left(n^{-1/2}\right),
\end{aligned} \tag{60}$$

where the triangular arrays $\{\xi_{j,n}\}$ and $\{\xi'_{j,n}\}$ are defined as

$$\begin{aligned}
\xi_{j,n} &\doteq n^{1/2} \left(n^{1/2} \sigma^3_{\tau_{j-1,n}} \left(\Delta_j^n W\right)^3 - (a-b) \, \sigma^3_{\tau_{j-1,n}} \Delta_{j,n} + 3\, n^{1/2} \sigma^2_{\tau_{j-1,n}} \left(\Delta_j^n W\right)^2 R_{j,n} \right), \\
\xi'_{j,n} &\doteq -(a-b) \, n^{1/2} \int_{\tau_{j-1,n}}^{\tau_{j,n}} \left(\sigma_s^3 - \sigma^3_{\tau_{j-1,n}}\right) ds,
\end{aligned}$$

The term in $\xi_{j,n}$ containing the remainder $R_{j,n}$ can now be expanded as follows

$$\begin{aligned}
&\left(\Delta_j^n W\right)^2 \left(R_{j,n} - \zeta_{j,n}(1)\right) \\
&= \left(\Delta_j^n W\right)^2 \int_{\tau_{j-1,n}}^{\tau_{j,n}} \mu_{\tau_{j-1,n}}\, du \\
&+ \left(\Delta_j^n W\right)^2 \left(\sigma'_{\tau_{j-1,n}} \int_{\tau_{j-1,n}}^{\tau_{j,n}} \left(W_s - W_{\tau_{j-1,n}}\right) dW_s + \nu_{\tau_{j-1,n}} \int_{\tau_{j-1,n}}^{\tau_{j,n}} \left(V_s - V_{\tau_{j-1,n}}\right) dW_s \right) \\
&= \mu_{\tau_{j-1,n}} \left(\Delta_j^n W\right)^2 \Delta_{j,n} + \left(\Delta_j^n W\right)^2 \left(\frac{\sigma'_{\tau_{j-1,n}}}{2} \left(\left(\Delta_j^n W\right)^2 - \Delta_{j,n} \right) + \nu_{\tau_{j-1,n}} \int_{\tau_{j-1,n}}^{\tau_{j,n}} \left(V_s - V_{\tau_{j-1,n}}\right) dW_s \right) \\
&= \mu_{\tau_{j-1,n}} \left(\Delta_j^n W\right)^2 \Delta_{j,n} + \frac{\sigma'_{\tau_{j-1,n}}}{2} \left(\left(\Delta_j^n W\right)^4 - \Delta_{j,n} \left(\Delta_j^n W\right)^2 \right) \\
&+ \nu_{\tau_{j-1,n}} \left(\Delta_j^n W\right)^2 \int_{\tau_{j-1,n}}^{\tau_{j,n}} \left(V_s - V_{\tau_{j-1,n}}\right) dW_s.
\end{aligned}$$

Using the fourth Bartlett identity and the fact that $\mathbb{E}_{\tau_{j-1,n}}[(\Delta_j^n W)^2 \int_{\tau_{j-1,n}}^{\tau_{j,n}} (V_s - V_{\tau_{j-1,n}})\, dW_s] = 0$ (independence of V from W and the sampling times, in addition to the tower rule), we obtain

$$\begin{aligned}
&\mathbb{E}_{\tau_{j-1,n}}\left[\left(\Delta_j^n W\right)^2 R_{j,n}\right] \\
&= \mathbb{E}_{\tau_{j-1,n}}\left[\left(\Delta_j^n W\right)^2 \zeta_{j,n}(1)\right] + \mu_{\tau_{j-1,n}}\, \mathbb{E}_{\tau_{j-1,n}}\left[\left(\Delta_j^n W\right)^2 \Delta_{j,n}\right] + \frac{\sigma'_{\tau_{j-1,n}}}{2}\, \mathbb{E}_{\tau_{j-1,n}}\left[\left(\Delta_j^n W\right)^4 - \Delta_{j,n}\left(\Delta_j^n W\right)^2\right] \\
&= \mathbb{E}_{\tau_{j-1,n}}\left[\left(\Delta_j^n W\right)^2 \zeta_{j,n}(1)\right] + \mu_{\tau_{j-1,n}}\, \mathbb{E}_{\tau_{j-1,n}}\left[\frac{1}{6}\left(\Delta_j^n W\right)^4 + \frac{1}{2}\Delta_{j,n}^2\right] + \frac{\sigma'_{\tau_{j-1,n}}}{2}\, \mathbb{E}_{\tau_{j-1,n}}\left[\frac{5}{6}\left(\Delta_j^n W\right)^4 - \frac{1}{2}\Delta_{j,n}^2\right].
\end{aligned}$$

Note that $\zeta_{j,n}(1) = o_p(n^{-1})$ and, thus,

$$\mathbb{E}_{\tau_{j-1,n}}\left[\left(\Delta_j^n W\right)^2 R_{j,n}\right] = \mu_{\tau_{j-1,n}}\, \mathbb{E}_{\tau_{j-1,n}}\left[\frac{1}{6}\left(\Delta_j^n W\right)^4 + \frac{1}{2}\Delta_{j,n}^2\right] + \frac{\sigma'_{\tau_{j-1,n}}}{2}\, \mathbb{E}_{\tau_{j-1,n}}\left[\frac{5}{6}\left(\Delta_j^n W\right)^4 - \frac{1}{2}\,\Delta_{j,n}^2\right] + o_p\left(n^{-2}\right). \quad (61)$$

Given the fourth result in Eq. (53) in addition to Eq (57) and Eq. (61), we have

$$\begin{aligned}
&\sum_{\tau_{j,n}\le t} \mathbb{E}_{\tau_{j-1,n}}\left[\xi_{j,n}\right] \\
&= \sum_{\tau_{j,n}\le t} \left(\sigma^3_{\tau_{j-1,n}} \left(n^{-1/2}\, a\, b\, (a-b) - n^{-1/2}\, a\, b\, (a-b)\right) + 3\, n\, \sigma^2_{\tau_{j-1,n}}\, \mathbb{E}_{\tau_{j-1,n}}\left[\left(\Delta_j^n W\right)^2 R_{j,n}\right] \right) \\
&= \sum_{\tau_{j,n}\le t} 3\, n\, \sigma^2_{\tau_{j-1,n}}\, \mathbb{E}_{\tau_{j-1,n}}\left[\left(\Delta_j^n W\right)^2 R_{j,n}\right] \\
&= \sum_{\tau_{j,n}\le t} 3\, n\, \sigma^2_{\tau_{j-1,n}}\, \mu_{\tau_{j-1,n}}\, \mathbb{E}_{\tau_{j-1,n}}\left[\frac{1}{6}\left(\Delta_j^n W\right)^4 + \frac{1}{2}\,\Delta_{j,n}^2\right] + \frac{3\,n}{2} \sum_{\tau_{j,n}\le t} \sigma^2_{\tau_{j-1,n}}\, \sigma'_{\tau_{j-1,n}}\, \mathbb{E}_{\tau_{j-1,n}}\left[\frac{5}{6}\left(\Delta_j^n W\right)^4 - \frac{1}{2}\,\Delta_{j,n}^2\right] \quad (62) \\
&= 3\left(\frac{1}{6} a\, b\, \left(a^2 - a\, b + b^2\right) + \frac{a\, b\, \left(a^2 + 3\, a\, b + b^2\right)}{6}\right) \frac{N_{t,n}}{t\, n} \sum_{\tau_{j,n}\le t} \sigma^2_{\tau_{j-1,n}}\, \mu_{\tau_{j-1,n}}\, \frac{t}{N_{t,n}} \\
&+ \frac{3}{2}\left(\frac{5}{6} a\, b\, \left(a^2 - a\, b + b^2\right) - \frac{ab\left(a^2 + ab + b^2\right)}{6}\right) \frac{N_{t,n}}{t\, n} \sum_{\tau_{j,n}\le t} \sigma^2_{\tau_{j-1,n}}\, \sigma'_{\tau_{j-1,n}}\, \frac{t}{N_{t,n}} \\
&\xrightarrow{p} \frac{1}{3}\left(a^2 + a\, b + b^2\right) \int_0^t \sigma_s^2\, \mu_s\, ds + \frac{1}{2}\left(2a^2 - 3ab + 2b^2\right) \int_0^t \sigma_s^2\, \sigma'_s\, ds,
\end{aligned}$$

where we have used the convergence in Eq. (52). Next, we turn to the contribution to the limiting distribution generated by $\{\xi'_{j,n}\}$. Itô's lemma for σ^3 gives

$$d\sigma_u^3 = 3\,\sigma_u^2\, \left(\mu'_u\, du + \sigma'_u\, dW_u + \nu_u\, dV_u\right) + 3\,\sigma_u\, \left(\sigma'^2_u + \nu_u^2\right) du.$$

Since V and W are independent, it follows that all the sampling times are independent of V. Thus, we have

$$\begin{aligned}
&\sum_{\tau_{j,n}\le t} \mathbb{E}_{\tau_{j-1,n}}\left[\xi'_{j,n}\right] \\
&= -3\,(a-b)\,n^{1/2} \sum_{\tau_{j,n}\le t} \mathbb{E}_{\tau_{j-1,n}}\left[\int_{\tau_{j-1,n}}^{\tau_{j,n}}\left(\int_{\tau_{j-1,n}}^{s} \sigma_u^2\,\sigma'_u\,dW_u\right)ds\right] + O_p\left(n^{-1/2}\right) \\
&= -3\,(a-b)\,n^{1/2} \sum_{\tau_{j,n}\le t} \mathbb{E}_{\tau_{j-1,n}}\left[\int_{\tau_{j-1,n}}^{\tau_{j,n}}\left(\int_{\tau_{j-1,n}}^{s} \left(\sigma^2_{\tau_{j-1,n}}\,\sigma'_{\tau_{j-1,n}} + O_p\left(n^{-1/2}\right)\right)dW_u\right)ds\right] \\
&+ O_p\left(n^{-1/2}\right) \\
&= -3\,(a-b)\,n^{1/2} \sum_{\tau_{j,n}\le t} \sigma^2_{\tau_{j-1,n}}\,\sigma'_{\tau_{j-1,n}}\,\mathbb{E}_{\tau_{j-1,n}}\left[\int_{\tau_{j-1,n}}^{\tau_{j,n}}\left(W_s - W_{\tau_{j-1,n}}\right)ds\right] + O_p\left(n^{-1/2}\right) \\
&= -\,(a-b)\,n^{1/2} \sum_{\tau_{j,n}\le t} \sigma^2_{\tau_{j-1,n}}\,\sigma'_{\tau_{j-1,n}}\,\mathbb{E}_{\tau_{j-1,n}}\left[\left(\Delta_j^n W\right)^3\right] + O_p\left(n^{-1/2}\right),
\end{aligned}$$

where the last equation results from Eq. (44) and the third Bartlett identity. In summary, using Lemma 7, we get

$$\sum_{\tau_{j,n}\le t} \mathbb{E}_{\tau_{j-1,n}}\left[\xi'_{j,n}\right] = -a\,b\,(a-b)^2\,\frac{1}{n} \sum_{\tau_{j,n}\le t} \sigma^2_{\tau_{j-1,n}}\,\sigma'_{\tau_{j-1,n}} + O_p\left(n^{-1/2}\right) \xrightarrow{p} -(a-b)^2 \int_0^t \sigma_s^2\,\sigma'_s\,ds. \tag{63}$$

Combining the convergence in Eq. (62) and that in Eq. (63) we obtain the coefficient $c_1(a,b) = a\,b/2$ that appears in the statement. To verify the second condition in Theorem 1 of Podolskij and Vetter (2010), we first notice that

$$\xi'^2_{j,n} = O_p\left(n^{-2}\right), \quad \xi_{j,n}\xi'_{j,n} = O_p\left(n^{-3/2}\right).$$

Hence, only $\xi^2_{j,n}$ contributes to the asymptotic variance. Accordingly, compute

$$\begin{aligned}
&\sum_{\tau_{j,n}\le t} \mathbb{E}_{\tau_{j-1,n}}\left[\xi^2_{j,n}\right] \\
&= \sum_{\tau_{j,n}\le t} \sigma^6_{\tau_{j-1,n}}\,\mathbb{E}_{\tau_{j-1,n}}\left[n^2\,\left(\Delta_j^n W\right)^6 + n\,(a-b)^2\,\Delta^2_{j,n} - 2\,n^{3/2}\,(a-b)\left(\Delta_j^n W\right)^3\,\Delta_{j,n}\right] \\
&+ O_p\left(\frac{1}{n^{1/2}}\right) \\
&= \sum_{\tau_{j,n}\le t} \sigma^6_{\tau_{j-1,n}}\left(\frac{ab\left(a^5+b^5\right)}{n\,(a+b)} + (a-b)^2\,\frac{ab\left(a^2+3ab+b^2\right)}{3n} - 2\,\frac{ab(a-b)^2(a+b)^2}{3n}\right) \\
&+ O_p\left(\frac{1}{n^{1/2}}\right),
\end{aligned}$$

by the cross moment in Eq. (56). Since $\sum_{\tau_{j,n}\le t} \mathbb{E}_{\tau_{j-1,n}}[\xi_{j,n}]^2 \xrightarrow{p} 0$, we obtain

$$\begin{aligned}
\sum_{\tau_{j,n}\le t}\left(\mathbb{E}_{\tau_{j-1,n}}\left[\xi^2_{j,n}\right] - \mathbb{E}_{\tau_{j-1,n}}\left[\xi_{j,n}\right]^2\right) &\xrightarrow{p} \left(\frac{\left(a^5+b^5\right)}{a+b} + \frac{(a-b)^2\left(a^2+3ab+b^2\right)}{3} - 2\,\frac{(a-b)^2(a+b)^2}{3}\right)\int_0^t \sigma_s^6\,ds \\
&= \frac{1}{3}\left(2a^4 - 2a^3b + 3a^2b^2 - 2ab^3 + 2b^4\right)\int_0^t \sigma_s^6\,ds \doteq \int_0^t \left(q_s^2 + u_s^2\right)ds,
\end{aligned}$$

where the last identity is definitional for the processes $\{q_s\}$ and $\{u_s\}$. The third condition is verified by computing

$$\begin{aligned}
&\sum_{\tau_{j,n}\le t} \mathbb{E}_{\tau_{j-1,n}}\left[\xi_{j,n}\,\Delta_j^n W\right] = \\
&= n^{1/2}\sum_{\tau_{j,n}\le t} \mathbb{E}_{\tau_{j-1,n}}\left[\left(n^{1/2}\,\sigma^3_{\tau_{j-1,n}}\left(\Delta_j^n W\right)^3 - (a-b)\,\sigma^3_{\tau_{j-1,n}}\,\Delta_{j,n} + 3\,n^{1/2}\,\sigma^2_{\tau_{j-1,n}}\left(\Delta_j^n W\right)^2 R_{j,n}\right)\Delta_j^n W\right] = \\
&= n^{1/2}\sum_{\tau_{j,n}\le t} \mathbb{E}_{\tau_{j-1,n}}\left[n^{1/2}\,\sigma^3_{\tau_{j-1,n}}\left(\Delta_j^n W\right)^4 - (a-b)\,\sigma^3_{\tau_{j-1,n}}\,\Delta_{j,n}\,\Delta_j^n W\right] + O_p\left(\frac{1}{n^{1/2}}\right), \qquad (64)
\end{aligned}$$

where we have used $R_{j,n} = O_p(n^{-1})$. Using the third Bartlett identity, we obtain

$$\begin{aligned}
\sum_{\tau_{j,n}\le t} \mathbb{E}_{\tau_{j-1,n}}\left[\xi_{j,n}\,\Delta_j^n W\right] &= n^{1/2}\sum_{\tau_{j,n}\le t} \mathbb{E}_{\tau_{j-1,n}}\left[n^{1/2}\,\sigma^3_{\tau_{j-1,n}}\left(\Delta_j^n W\right)^4 - \frac{a-b}{3}\sigma^3_{\tau_{j-1,n}}\left(\Delta_j^n W\right)^3\right] \\
&= n^{1/2}\sum_{\tau_{j,n}\le t}\left(n^{1/2}\,\sigma^3_{\tau_{j-1,n}}\,n^{-2}\,a\,b\,\left(a^2-ab+b^2\right) - \frac{1}{3}\sigma^3_{\tau_{j-1,n}}\,n^{-3/2}\,a\,b\,(a-b)^2\right) \\
&\xrightarrow{p} \left(a^2-ab+b^2\right)\int_0^t \sigma_s^3\,ds - \frac{(a-b)^2}{3}\int_0^t \sigma_s^3\,ds \\
&= \frac{1}{3}\left(2a^2-ab+2b^2\right)\int_0^t \sigma_s^3\,ds = \int_0^t q_s\,ds,
\end{aligned}$$

which, as a consequence, yields

$$u_s^2 = \frac{1}{3}\left(2a^4-2a^3b+3a^2b^2-2ab^3+2b^4\right)\sigma_s^6 - \frac{1}{9}\left(2a^2-ab+2b^2\right)^2\sigma_s^6 = \frac{2}{9}(a-b)^2\left(a^2+ab+b^2\right)\sigma_s^6.$$

Finally, it is not difficult to prove that

$$\sum_{\tau_{j,n}\le t} \mathbb{E}_{\tau_{j-1,n}}\left[\xi_{j,n}^4\right] \xrightarrow{p} 0.$$

In order to obtain stable convergence, we now employ Assumption 2 explicitly. Let N be a $\mathcal{F}_t$-martingale such that $N_0 = 0$ and $[W,N] \equiv 0$. First, note that

$$\begin{aligned}
\mathbb{E}_{\tau_{j-1,n}}\left[\left(\Delta_j^n W\right)^3\Delta_j^n N\right] &= 3\,\mathbb{E}_{\tau_{j-1,n}}\left[\left(\int_{\tau_{j-1,n}}^{\tau_{j,n}}\left(W_s - W_{\tau_{j-1,n}}\right)^2 dW_s + \int_{\tau_{j-1,n}}^{\tau_{j,n}}\left(W_s - W_{\tau_{j-1,n}}\right)ds\right)\int_{\tau_{j-1,n}}^{\tau_{j,n}} dN_s\right] \\
&= 3\,\mathbb{E}_{\tau_{j-1,n}}\left[\int_{\tau_{j-1,n}}^{\tau_{j,n}}\left(\left(W_s - W_{\tau_{j-1,n}}\right)^2 + \left(\tau_{j,n}-s\right)\right)dW_s\int_{\tau_{j-1,n}}^{\tau_{j,n}} dN_s\right] \\
&= 3\,\mathbb{E}_{\tau_{j-1,n}}\left[\int_{\tau_{j-1,n}}^{\tau_{j,n}}\left(\left(W_s - W_{\tau_{j-1,n}}\right)^2 + \left(\tau_{j,n}-s\right)\right)d[W,N]_s\right] = 0,
\end{aligned}$$

where the first equality follows from Itô's lemma and the second from Fubini's theorem. Second, the Kunita-Watanabe representation implies that

$$N_t = \sum_{k=0}^{\mathrm{p}}\int_0^t \eta_s^{(k)}\,dW_s^{(k)},$$

where we have used the notation $W^{(0)} \equiv V$ and where the $\{\eta_t^{(k)}\}$s are suitable processes. Take any $j \ge 1$ and consider the σ-algebra

$$\mathcal{H} \doteq \mathcal{F}_{\tau_{j-1,n}} \vee \sigma\left\{W_{\tau_{j-1,n}+s} - W_{\tau_{j-1,n}} \;\middle|\; s \ge 0\right\}.$$

Naturally, $\mathcal{F}_{\tau_{j-1,n}} \subset \mathcal{H}$. We also have that $\tau_{j-1,n}$ and $\tau_{j,n}$ are both $\mathcal{H}$-measurable and, thus, so is $\Delta_{j,n} \in \mathcal{H}$. Also,

$$\mathbb{E}\left[\int_{\tau_{j-1,n}}^{\tau_{j,n}} \eta_s^{(k)}\,dW_s^{(k)} \;\middle|\; \mathcal{H}\right] = 0, \quad k = 0,\cdots,p.$$

Finally, the $\mathcal{F}_{\tau_{j-1,n}}$-conditional expected value of the product between the j-th time increment and the corresponding

N increment is

$$\mathbb{E}\left[\Delta_{j,n}\,\Delta_j^n N \,\middle|\, \mathcal{F}_{\tau_{j-1,n}}\right] = \mathbb{E}\left[\mathbb{E}\left[\Delta_{j,n}\,\Delta_j^n N \,\middle|\, \mathcal{H}\right] \,\middle|\, \mathcal{F}_{\tau_{j-1,n}}\right] = \sum_{k=0}^{\mathrm{p}} \mathbb{E}\left[\Delta_{j,n}\,\mathbb{E}\left[\int_{\tau_{j-1,n}}^{\tau_{j,n}} \eta_s^{(k)}\, dW_s^{(k)} \,\middle|\, \mathcal{H}\right] \,\middle|\, \mathcal{F}_{\tau_{j-1,n}}\right] = 0.$$

It is now immediate to see that $\mathbb{E}[R_{j,n}\Delta_j^n N \mid \mathcal{F}_{\tau_{j-1,n}}] = 0$. Hence,

$$\sum_{\tau_{j,n}\le t} \mathbb{E}_{\tau_{j-1,n}}\left[\xi_{j,n}\,\Delta_j^n N\right] = 0,$$

which completes the proof. □

Lemma 8. *Let Assumptions 1, 2 and 4 hold. For each $j \ge 1$, consider the event $A_{j,n} \doteq \{X_{\tau_{j,n}} - X_{\tau_{j-1,n}} = U_n\}$. We have the following expansions*

$$\begin{aligned}
\mathbb{P}\left[A_{j,n} \,\middle|\, \mathcal{F}_{\tau_{j-1,n}}\right] &= \frac{b}{a+b} + \frac{\mu_{\tau_{j-1,n}}}{\sigma^2_{\tau_{j-1,n}}}\,\frac{ab}{(a+b)\sqrt{n}} + O_p\left(\frac{1}{n}\right),\\
\mathbb{E}\left[\tau_{j,n} - \tau_{j-1,n} \,\middle|\, \mathcal{F}_{\tau_{j-1,n}}\right] &= \frac{ab}{n\,\sigma^2_{\tau_{j-1,n}}} + \frac{\mu_{\tau_{j-1,n}}\,a\,b\,(a-b)}{3n^{3/2}\,\sigma^4_{\tau_{j-1,n}}} + O_p\left(\frac{1}{n^2}\right).
\end{aligned} \tag{65}$$

Proof. To simplify notation, for a given j, let $\tau_0 \doteq \tau_{j-1,n}$. Consider the process $Z_s \doteq X_{\tau_0+s} - X_{\tau_0}$, for $s \ge 0$. The process $W_s^\star \doteq W_{\tau_0+s} - W_{\tau_0}$ is a Brownian motion with respect to the filtration $\mathcal{F}_s^\star \doteq \mathcal{F}_{\tau_0+s}$. We have

$$Z_s = \int_{\tau_0}^{\tau_0+s} \mu_u\, du + \int_{\tau_0}^{\tau_0+s} \sigma_u\, dW_u = \int_0^s \mu_{\tau_0+u}\, du + \int_0^s \sigma_{\tau_0+u}\, dW_u^\star,$$

which implies that Z solves

$$\begin{cases} Z_0 = 0 \\ dZ_s = \mu_{\tau_0+s}\, ds + \sigma_{\tau_0+s}\, dW_s^\star. \end{cases}$$

Let $\widetilde{Z}$ be the solution of the same differential equation with constant coefficients, that is

$$\begin{cases} \widetilde{Z}_0 = x \in (-\ell_n, U_n) \\ d\widetilde{Z}_s = \mu_{\tau_0}\, ds + \sigma_{\tau_0}\, dW_s^\star. \end{cases}$$

In this proof, $\mathbb{E}_x[\cdot]$ and $\mathbb{P}[\cdot \mid \widetilde{Z}_0 = x]$ refer to the frozen process started at x, conditionally on $\mathcal{F}_{\tau_0}$. Define the exit time $\tau_1 = \inf\{t > 0 : \widetilde{Z}_t \notin \mathcal{U}_n\}$. Also, $\tau^+ \doteq \inf\{t > 0 : \widetilde{Z}_t = +U_n\}$ and $\tau^- \doteq \inf\{t > 0 : \widetilde{Z}_t = -\ell_n\}$, so that $\tau_1 = \tau^+ \wedge \tau^-$. Define the function $u_n(x) \doteq \mathbb{P}[\tau^+ < \tau^- \mid \widetilde{Z}_0 = x]$. Note that $u_n(+U_n) = 1$ and $u_n(-\ell_n) = 0$. Moreover,

$$\widetilde{Z}_{\tau_1} = \begin{cases} +U_n & \text{on } \{\tau^+ < \tau^-\} \\ -\ell_n & \text{on } \{\tau^- < \tau^+\}. \end{cases}$$

Thus, $u_n(\widetilde{Z}_{\tau_1}) = \mathbb{1}_{\{\tau^+<\tau^-\}}$, which means $u_n(x) = \mathbb{P}[\tau^+ < \tau^- \mid \widetilde{Z}_0 = x] = \mathbb{E}_x[\mathbb{1}_{\{\tau^+<\tau^-\}}] = \mathbb{E}_x[u_n(\widetilde{Z}_{\tau_1})]$. The infinitesimal generator of $\widetilde{Z}$ is

$$(\mathcal{L}f)(x) = \mu_{\tau_0} f'(x) + \frac{1}{2}\sigma^2_{\tau_0} f''(x).$$

Hence, Dynkin's formula applied to the function $u_n(x)$ gives

$$\mathbb{E}_x\left[u_n\left(\widetilde{Z}_{\tau_1}\right)\right] - u_n(x) = \mathbb{E}_x\left[\int_0^{\tau_1} (\mathcal{L}u_n)\left(\widetilde{Z}_s\right)\, ds\right],$$

which, since $\mathbb{E}_x[u_n(\widetilde{Z}_{\tau_1})] = u_n(x)$, implies $\mathbb{E}_x[\int_0^{\tau_1}(\mathcal{L}u_n)(\widetilde{Z}_s)\,ds] = 0$. The latter, in turn, implies that u_n solves the differential equation

$$\begin{cases} u_n(+U_n) = 1, u_n(-\ell_n) = 0 \\ (\mathcal{L}u_n)(x) = 0, \end{cases}$$

whose solution is

$$u_n(x) = \frac{1 - e^{-\alpha_{\tau_0}(x+\ell_n)}}{1 - e^{-\alpha_{\tau_0}(U_n+\ell_n)}},$$

with $\alpha_{\tau_0} \doteq 2\,\mu_{\tau_0}/\sigma^2_{\tau_0}$. Hence, we have the following

$$u_n(0) = \frac{1 - e^{-\alpha_{\tau_0} b/\sqrt{n}}}{1 - e^{-\alpha_{\tau_0}(a+b)/\sqrt{n}}} = \frac{b}{a+b} + \frac{\mu_{\tau_0}}{\sigma^2_{\tau_0}} \frac{ab}{(a+b)\sqrt{n}} + O_p\left(\frac{1}{n}\right),$$

as claimed. Finally, since $\widetilde{Z}$ has contant drift μ_{τ_0}, then $\widetilde{Z}_t - \mu_{\tau_0} t$ is a martingale and, thus, from

$$\mathbb{E}_0\left[\widetilde{Z}_{\tau_1} - \widetilde{Z}_0\right] = \mu_{\tau_0}\,\mathbb{E}_0[\tau_1]$$

we derive the following expansion for $\mathbb{E}_0[\tau_1]$:

$$\mathbb{E}_0[\tau_1] = \frac{1}{\mu_{\tau_0}}\,\mathbb{E}_0\left[\widetilde{Z}_{\tau_1} - \widetilde{Z}_0\right] = \frac{a\,u_n(0) - b\,(1 - u_n(0))}{\sqrt{n}\,\mu_{\tau_0}} = \frac{ab}{n\,\sigma^2_{\tau_0}} + \frac{\mu_{\tau_0} a\,b\,(a-b)}{3n^{3/2}\,\sigma^4_{\tau_0}} + O\left(\frac{1}{n^2}\right).$$

□

Corollary A.1. *Under the hypotheses of Lemma 8, the following moments expansions apply:*

$$\begin{aligned}
\mathbb{E}_{\tau_{j-1,n}}\left[X_{\tau_{j,n}} - X_{\tau_{j-1,n}}\right] &= \frac{a\,b}{n}\frac{\mu_{\tau_{j-1,n}}}{\sigma^2_{\tau_{j-1,n}}} + O_p\left(\frac{1}{n^{3/2}}\right),\\
\mathbb{E}_{\tau_{j-1,n}}\left[\left(X_{\tau_{j,n}} - X_{\tau_{j-1,n}}\right)^2\right] &= \frac{a\,b}{n} + \frac{a\,b\,(a-b)}{n^{3/2}}\frac{\mu_{\tau_{j-1,n}}}{\sigma^2_{\tau_{j-1,n}}} + O_p\left(\frac{1}{n^2}\right),\\
\mathbb{E}_{\tau_{j-1,n}}\left[\left(X_{\tau_{j,n}} - X_{\tau_{j-1,n}}\right)^3\right] &= \frac{ab(a-b)}{n^{3/2}} + \frac{a\,b\,\left(a^2 - ab + b^2\right)}{n^2}\frac{\mu_{\tau_{j-1,n}}}{\sigma^2_{\tau_{j-1,n}}} + O_p\left(\frac{1}{n^{5/2}}\right),\\
\mathbb{E}_{\tau_{j-1,n}}\left[\left(X_{\tau_{j,n}} - X_{\tau_{j-1,n}}\right)^4\right] &= \frac{ab\left(a^2 - ab + b^2\right)}{n^2} + \frac{a\,b\,(a-b)\left(a^2+b^2\right)}{n^{5/2}}\frac{\mu_{\tau_{j-1,n}}}{\sigma^2_{\tau_{j-1,n}}} + O_p\left(\frac{1}{n^3}\right),\\
\mathbb{E}_{\tau_{j-1,n}}\left[\left(X_{\tau_{j,n}} - X_{\tau_{j-1,n}}\right)^5\right] &= \frac{a\,b\,(a-b)\left(a^2+b^2\right)}{n^{5/2}} + \frac{a\,b\,\left(a^5+b^5\right)}{n^3\,(a+b)}\frac{\mu_{\tau_{j-1,n}}}{\sigma^2_{\tau_{j-1,n}}} + O_p\left(\frac{1}{n^{7/2}}\right),\\
\mathbb{E}_{\tau_{j-1,n}}\left[\left(X_{\tau_{j,n}} - X_{\tau_{j-1,n}}\right)^6\right] &= \frac{ab\left(a^5+b^5\right)}{n^3(a+b)} + \frac{a\,b\,\left(a^6 - b^6\right)}{n^{7/2}\,(a+b)}\frac{\mu_{\tau_{j-1,n}}}{\sigma^2_{\tau_{j-1,n}}} + O_p\left(\frac{1}{n^4}\right).
\end{aligned}$$

Proof. All of the expansions are an immediate consequence of the expression in Eq. (65). □

Proof of Lemma 2. Fix $j \geq 1$ and, for $t \geq \tau_{j-1,n}$, let $\delta^n_j X(t) \doteq X_t - X_{\tau_{j-1,n}}$. Thus, $\delta^n_j X(\tau_{j-1,n}) = 0$. Also, $\delta^n_j X(\tau_{j,n}) = +U_n$ or $\delta^n_j X(\tau_{j,n}) = -\ell_n$ and $\delta^n_j X$ is continuous as X. Like in the proof of Lemma 1, it is easy to see that $\delta^n_j X(t) = O_p(n^{-1/2})$, uniformly on $[\tau_{j-1,n}, \tau_{j,n}]$. Consider, for all $n \in \mathbb{N}$, the function

$$u^{(n)}(y,\sigma) \doteq \frac{1}{\sigma^2}\,(U_n - y)\,(\ell_n + y) = \frac{1}{\sigma^2}\left(U_n\,\ell_n + (U_n - \ell_n)\,y - y^2\right),$$

which has the following first and second order partial derivatives:

$$\begin{cases}
u^{(n)}_y = (U_n - \ell_n - 2\,y)/\sigma^2,\\
u^{(n)}_\sigma = -2\,(U_n - y)\,(\ell_n + y)/\sigma^3,\\
u^{(n)}_{yy} = -2/\sigma^2,\\
u^{(n)}_{\sigma\sigma} = 6\,(U_n - y)\,(\ell_n + y)/\sigma^4,\\
u^{(n)}_{y\sigma} = -2\,(U_n - \ell_n - 2\,y)/\sigma^3.
\end{cases} \tag{66}$$

We also have the following boundary conditions:

$$\begin{cases} u^{(n)}\left(\delta_j^n X\left(\tau_{j,n}\right), \sigma_{\tau_{j,n}}\right) = 0 \\ u^{(n)}\left(\delta_j^n X\left(\tau_{j-1,n}\right), \sigma_{\tau_{j-1,n}}\right) = u^{(n)}\left(0, \sigma_{\tau_{j-1,n}}\right) = a\, b/(n\, \sigma^2_{\tau_{j-1,n}}). \end{cases} \tag{67}$$

Itô's lemma for $Z_t \doteq u^{(n)}(\delta_j^n X(t), \sigma_t)$, now, gives

$$\begin{aligned} dZ_t &= u_y^{(n)}\, d\left(\delta_j^n X\left(t\right)\right) + u_\sigma^{(n)}\, d\sigma_t + \frac{1}{2}\, u_{yy}^{(n)}\, d\left[\delta_j^n X\right]_t + \frac{1}{2}\, u_{\sigma\sigma}^{(n)}\, d\left[\sigma\right]_t + u_{y\sigma}^{(n)}\, d\left[\delta_j^n X, \sigma\right]_t \\ &= u_y^{(n)}\, \left(\mu_t\, dt + \sigma_t\, dW_t\right) + u_\sigma^{(n)}\, \left(\mu_t'\, dt + \sigma_t'\, dW_t + \nu_t\, dV_t\right) - dt + \frac{1}{2}\, u_{\sigma\sigma}^{(n)}\, \left(\sigma_t'^2 + \nu_t^2\right)\, dt + u_{y\sigma}^{(n)}\, \sigma_t\, \sigma_t'\, dt. \end{aligned}$$

Integrating from $\tau_{j-1,n}$ to $\tau_{j,n}$, and using the boundary conditions in Eq. (67), we obtain

$$\tau_{j,n} - \tau_{j-1,n} = \frac{ab}{n\, \sigma^2_{\tau_{j-1,n}}} + \int_{\tau_{j-1,n}}^{\tau_{j,n}} \Phi_{j,n}^{(W)}\left(t\right)\, dW_t + \int_{\tau_{j-1,n}}^{\tau_{j,n}} \Phi_{j,n}^{(V)}\left(t\right)\, dV_t + \int_{\tau_{j-1,n}}^{\tau_{j,n}} \Phi_{j,n}^{(\Delta)}\left(t\right)\, dt, \tag{68}$$

where

$$\Phi_{j,n}^{(W)}\left(t\right) \doteq u_y^{(n)}\, \sigma_t\, + u_\sigma^{(n)}\, \sigma_t', \quad \Phi_{j,n}^{(V)}\left(t\right) \doteq u_\sigma^{(n)}\, \nu_t, \quad \Phi_{j,n}^{(\Delta)}\left(t\right) \doteq u_y^{(n)}\, \mu_t + u_\sigma^{(n)}\, \mu_t' + \frac{1}{2}\, u_{\sigma\sigma}^{(n)}\, \left(\sigma_t'^2 + \nu_t^2\right) + u_{y\sigma}^{(n)}\, \sigma_t\, \sigma_t'.$$

Since, as mentioned earlier, $\delta_j^n X(t) = O_p(n^{-1/2})$ uniformly on $[\tau_{j-1,n}, \tau_{j,n}]$, from the partial derivatives in Eq. (66), we obtain the following probability orders:

$$\begin{aligned} u_y^{(n)}\left(\delta_j^n X\left(t\right), \sigma_t\right) &= \frac{1}{\sigma_t^2}\left(\frac{a-b}{\sqrt{n}} - 2\, \delta_j^n X\left(t\right)\right) = O_p\left(n^{-1/2}\right), \\ u_\sigma^{(n)}\left(\delta_j^n X\left(t\right), \sigma_t\right) &= -\frac{2}{\sigma_t^3}\left(\frac{a\, b}{n} + \frac{(a-b)}{\sqrt{n}}\, \delta_j^n X\left(t\right) - \left(\delta_j^n X\left(t\right)\right)^2\right) = O_p\left(n^{-1}\right), \\ u_{\sigma\sigma}^{(n)}\left(\delta_j^n X\left(t\right), \sigma_t\right) &= \frac{6}{\sigma_t^4}\left(\frac{a}{\sqrt{n}} - \delta_j^n X\left(t\right)\right)\left(\frac{b}{\sqrt{n}} + \delta_j^n X\left(t\right)\right) = O_p\left(n^{-1}\right), \\ u_{y\sigma}^{(n)}\left(\delta_j^n X\left(t\right), \sigma_t\right) &= -\frac{2}{\sigma_t^3}\left(\frac{a-b}{\sqrt{n}} - 2\, \delta_j^n X\left(t\right)\right) = O_p\left(n^{-1/2}\right), \end{aligned}$$

uniformly on $[\tau_{j-1,n}, \tau_{j,n}]$. Consequently, we have

$$\Phi_{j,n}^{(W)}\left(t\right) = O_p\left(n^{-1/2}\right), \quad \Phi_{j,n}^{(V)}\left(t\right) = O_p\left(n^{-1}\right), \quad \Phi_{j,n}^{(\Delta)}\left(t\right) = O_p\left(n^{-1/2}\right),$$

and, thus,

$$\int_{\tau_{j-1,n}}^{\tau_{j,n}} \Phi_{j,n}^{(W)}\left(t\right)\, dW_t = O_p\left(n^{-1}\right), \quad \int_{\tau_{j-1,n}}^{\tau_{j,n}} \Phi_{j,n}^{(V)}\left(t\right)\, dV_t = O_p\left(n^{-3/2}\right), \quad \int_{\tau_{j-1,n}}^{\tau_{j,n}} \Phi_{j,n}^{(\Delta)}\left(t\right)\, dt = O_p\left(n^{-3/2}\right).$$

We may now show that, for all $\mathcal{F}_t$-martingale N such that $[N, W] \equiv 0$, it holds that

$$\sqrt{n} \sum_{\tau_{j,n} \leq t} \mathbb{E}_{\tau_{j-1,n}}\left[\Delta_{j,n}\, \Delta_j^n N\right] \xrightarrow{p} 0.$$

The result will be employed in the proof of Theorem 3.2. Using the fact that N is a martingale and that $[W, N] \equiv 0$, the order term $O_p(n^{-1})$ disappears. In fact, we have

$$\begin{aligned} &\mathbb{E}_{\tau_{j-1,n}}\left[\Delta_{j,n}\, \Delta_j^n N\right] \\ &= \mathbb{E}_{\tau_{j-1,n}}\left[\int_{\tau_{j-1,n}}^{\tau_{j,n}} \Phi_{j,n}^{(W)}\left(t\right)\, d[W,N]_t\right] + \mathbb{E}_{\tau_{j-1,n}}\left[\left(\int_{\tau_{j-1,n}}^{\tau_{j,n}} u_\sigma^{(n)}\left(\delta_j^n X\left(t\right), \sigma_t\right)\, \nu_t\, dV_t + \int_{\tau_{j-1,n}}^{\tau_{j,n}} \Phi_{j,n}^{(\Delta)}\left(t\right)\, dt\right) \Delta_j^n N\right] \\ &= \mathbb{E}_{\tau_{j-1,n}}\left[\left(\int_{\tau_{j-1,n}}^{\tau_{j,n}} u_\sigma^{(n)}\left(\delta_j^n X\left(t\right), \sigma_t\right)\, \nu_t\, dV_t + \int_{\tau_{j-1,n}}^{\tau_{j,n}} \Phi_{j,n}^{(\Delta)}\left(t\right)\, dt\right) \Delta_j^n N\right], \end{aligned}$$

which implies

$$\sqrt{n} \sum_{\tau_{j,n} \le t} \mathbb{E}_{\tau_{j-1,n}} \left[\Delta_{j,n} \, \Delta_j^n N \right] = O_p \left(\sup_j \Delta_j^n N \right) \xrightarrow{p} 0, \tag{69}$$

where we have relied on the fact that in a filtration generated by finitely many Brownian motions, every (local) martingale is continuous. N, in fact, has continuous paths. □

Lemma 9. *Let Assumptions 1, 2 and 4 hold. Let g be a bounded càdlàg process. Therefore,*

$$\frac{1}{n} \sum_{\tau_{j,n} \le t} g_{\tau_{j-1,n}} \xrightarrow{p} \frac{1}{a\,b} \int_0^t g_s \, \sigma_s^2 \, ds.$$

In particular

$$\frac{1}{n} \sum_{\tau_{j,n} \le t} 1 = \frac{N_{t,n}}{n} \xrightarrow{p} \frac{1}{a\,b} \int_0^t \sigma_s^2 \, ds.$$

Proof. The proof is just an application of the more general Lemma 7. □

Proof of Theorem 3.2. In order to prove stable convergence, we again verify that the five criteria in Theorem 1 of Podolskij and Vetter (2010) are met. Before proceeding, we compute specific moments. Set an index $j \ge 1$. Decompose, for $t \ge \tau_{j-1,n}$, $X_t = X_{\tau_{j-1,n}} + A_t + M_t$ with $A_t = \int_{\tau_{j-1,n}}^t \mu_s \, ds$ and $M_t = \int_{\tau_{j-1,n}}^t \sigma_s \, dW_s$. From the third Bartlett identity applied to the martingale M, we have

$$\mathbb{E}_{\tau_{j-1,n}} \left[\left(\Delta_j^n X \right)^3 - \left(\Delta_j^n A \right)^3 - 3 \left(\Delta_j^n X \right)^2 \Delta_j^n A + 3 \, \Delta_j^n X \left(\Delta_j^n A \right)^2 - 3 \left(\int_{\tau_{j-1,n}}^{\tau_{j,n}} \sigma_s^2 \, ds \right) \left(\Delta_j^n X - \Delta_j^n A \right) \right] = 0.$$

Since $\sigma_t^2 = \sigma_{\tau_{j-1,n}}^2 + O_p(n^{-1/2})$, we have $\int_{\tau_{j-1,n}}^{\tau_{j,n}} \sigma_s^2 \, ds = \sigma_{\tau_{j-1,n}}^2 \, \Delta_{j,n} + O_p(n^{-3/2})$. This expression, combined with $\Delta_j^n A = O_p(n^{-1})$, gives

$$\mathbb{E}_{\tau_{j-1,n}} \left[\left(\Delta_j^n X \right)^3 \right] = 3 \, \mathbb{E}_{\tau_{j-1,n}} \left[\sigma_{\tau_{j-1,n}}^2 \, \Delta_{j,n} \, \Delta_j^n X \right] + O_p \left(n^{-2} \right).$$

Using the third equation in Corollary A.1, we obtain the following cross-moment:

$$\mathbb{E}_{\tau_{j-1,n}} \left[\Delta_{j,n} \, \Delta_j^n X \right] = \frac{ab(a-b)}{3 \, n^{3/2} \, \sigma_{\tau_{j-1,n}}^2} + O_p \left(n^{-2} \right). \tag{70}$$

Note that, using the expression $\Delta_j^n X = \sigma_{\tau_{j-1,n}} \, \Delta_j^n W + O_p(n^{-1})$, Eq. (70) implies

$$\mathbb{E}_{\tau_{j-1,n}} \left[\Delta_{j,n} \left(\sigma_{\tau_{j-1,n}} \, \Delta_j^n W + O_p \left(n^{-1} \right) \right) \right] = \frac{ab(a-b)}{3 \, n^{3/2} \, \sigma_{\tau_{j-1,n}}^2} + O_p \left(n^{-2} \right),$$

which yields the cross-moment

$$\mathbb{E}_{\tau_{j-1,n}} \left[\Delta_{j,n} \, \Delta_j^n W \right] = \frac{ab(a-b)}{3 \, n^{3/2} \, \sigma_{\tau_{j-1,n}}^3} + O_p \left(n^{-2} \right). \tag{71}$$

Now, using the decomposition $\Omega = \{\Delta_j^n X = U_n\} \cup \{\Delta_j^n X = -\ell_n\}$, we obtain the following system of equations:

$$\begin{aligned}
&\mathbb{E}_{\tau_{j-1,n}} \left[\Delta_{j,n} \right] \\
&= \mathbb{E}_{\tau_{j-1,n}} \left[\Delta_{j,n}; \Delta_j^n X = U_n \right] + \mathbb{E}_{\tau_{j-1,n}} \left[\Delta_{j,n}; \Delta_j^n X = -\ell_n \right] \\
&= \frac{ab}{n \, \sigma_{\tau_{j-1,n}}^2} + \frac{\mu_{\tau_{j-1,n}} a \, b \, (a-b)}{3 n^{3/2} \, \sigma_{\tau_{j-1,n}}^4} + O_p \left(n^{-2} \right), \\
&\mathbb{E}_{\tau_{j-1,n}} \left[\Delta_{j,n} \, \Delta_j^n X \right] \\
&= U_n \, \mathbb{E}_{\tau_{j-1,n}} \left[\Delta_{j,n}; \Delta_j^n X = U_n \right] - \ell_n \, \mathbb{E}_{\tau_{j-1,n}} \left[\Delta_{j,n}; \Delta_j^n X = -\ell_n \right] \\
&= \frac{ab(a-b)}{3 \, n^{3/2} \, \sigma_{\tau_{j-1,n}}^2} + O_p \left(n^{-2} \right),
\end{aligned} \tag{72}$$

where the right-hand side of the first equation comes from the expansion in Eq. (65), while the second derives from Eq. (70). Solving the system of equations, we have

$$\mathbb{E}_{\tau_{j-1,n}}\left[\Delta_{j,n}; \Delta_j^n X = U_n\right] = \frac{a\,b\,(a+2\,b)}{3\,n\,(a+b)\,\sigma^2_{\tau_{j-1,n}}} + O_p\left(n^{-3/2}\right) \tag{73}$$

and, similarly,

$$\mathbb{E}_{\tau_{j-1,n}}\left[\Delta_{j,n}; \Delta_j^n X = -\ell_n\right] = \frac{a\,b\,(2\,a+b)}{3\,n\,(a+b)\,\sigma^2_{\tau_{j-1,n}}} + O_p\left(n^{-3/2}\right). \tag{74}$$

From Eqs. (73)-(74), we may now compute

$$\begin{aligned}\mathbb{E}_{\tau_{j-1,n}}\left[\Delta_{j,n}\left(\Delta_j^n X\right)^2\right] &= U_n^2\,\mathbb{E}_{\tau_{j-1,n}}\left[\Delta_{j,n}; \Delta_j^n X = U_n\right] + \ell_n^2\,\mathbb{E}_{\tau_{j-1,n}}\left[\Delta_{j,n}; \Delta_j^n X = -\ell_n\right] \\ &= \frac{ab\left(a^2+ab+b^2\right)}{3n^2\,\sigma^2_{\tau_{j-1,n}}} + O_p\left(n^{-5/2}\right),\end{aligned} \tag{75}$$

and, also,

$$\mathbb{E}_{\tau_{j-1,n}}\left[\Delta_{j,n}\left(\Delta_j^n X\right)^3\right] = \frac{ab(a-b)(a+b)^2}{3n^{5/2}\,\sigma^2_{\tau_{j-1,n}}} + O_p\left(n^{-3}\right).$$

Next, consider the fourth Bartlett identity for the martingale M and write

$$\mathbb{E}_{\tau_{j-1,n}}\left[\left(\Delta_j^n M\right)^4\right] = 6\,\mathbb{E}_{\tau_{j-1,n}}\left[\Delta_j^n[M]\left(\Delta_j^n M\right)^2\right] - 3\,\mathbb{E}_{\tau_{j-1,n}}\left[\left(\Delta_j^n[M]\right)^2\right].$$

Using $M = X - A$ and the expansion $\Delta_j^n[M] = \sigma^2_{\tau_{j-1,n}}\,\Delta_{j,n} + O_p(n^{-3/2})$, we obtain

$$\mathbb{E}_{\tau_{j-1,n}}\left[\left(\Delta_j^n X\right)^4\right] = 6\,\sigma^2_{\tau_{j-1,n}}\,\mathbb{E}_{\tau_{j-1,n}}\left[\Delta_{j,n}\left(\Delta_j^n X\right)^2\right] - 3\,\sigma^4_{\tau_{j-1,n}}\,\mathbb{E}_{\tau_{j-1,n}}\left[\Delta_{j,n}^2\right] + O_p\left(n^{-5/2}\right).$$

Using, also, the fourth equation in Corollary A.1 and Eq. (75), we obtain

$$\mathbb{E}_{\tau_{j-1,n}}\left[\Delta_{j,n}^2\right] = \frac{a\,b}{3\,\sigma^4_{\tau_{j-1,n}}\,n^2}\left(a^2+3\,a\,b+b^2\right) + O_p\left(n^{-5/2}\right).$$

We can now proceed to verify the conditions of Theorem 1 in Podolskij and Vetter (2010). From Corollary A.1, we have

$$n^{1/2}\sum_{\tau_{j,n}\le t}\mathbb{E}_{\tau_{j-1,n}}\left[\left(X_{\tau_{j,n}} - X_{\tau_{j-1,n}}\right)^3\right] = a\,b\,(a-b)\,\frac{N_{t,n}}{n} + O_p\left(\frac{1}{n^{1/2}}\right) \xrightarrow{p} (a-b)\int_0^t \sigma_s^2\,ds.$$

To complete the proof, we consider the estimation error decomposition

$$\begin{aligned} n^{1/2}\left(\sum_{\tau_{j,n}\le t} n^{1/2}\left(\Delta_j^n X\right)^3 - (a-b)\int_0^t \sigma_s^2\,ds\right) &= \sum_{\tau_{j,n}\le t}\xi_{j,n} + n^{1/2}\,(a-b)\left(\sum_{\tau_{j,n}\le t}\sigma^2_{\tau_{j-1,n}}\,\Delta_{j,n} - \int_0^t \sigma_s^2\,ds\right), \\ &= \sum_{\tau_{j,n}\le t}\xi_{j,n} + \sum_{\tau_{j,n}\le t}\xi'_{j,n} - (a-b)\,n^{1/2}\int_{\tau_{N_{t,n},n}}^t \sigma_s^2\,ds, \end{aligned}$$

where the triangular arrays $\{\xi_{j,n}\}$ and $\{\xi'_{j,n}\}$ are defined as

$$\begin{aligned}\xi_{j,n} &\doteq n^{1/2}\left(n^{1/2}\left(\Delta_j^n X\right)^3 - (a-b)\,\sigma^2_{\tau_{j-1,n}}\,\Delta_{j,n}\right), \\ \xi'_{j,n} &\doteq -n^{1/2}\,(a-b)\int_{\tau_{j-1,n}}^{\tau_{j,n}}\left(\sigma_s^2 - \sigma^2_{\tau_{j-1,n}}\right)ds.\end{aligned}$$

We have

$$\begin{aligned}\mathbb{E}_{\tau_{j-1,n}}\left[\xi_{j,n}\right] &= n\left(\frac{ab(a-b)}{n^{3/2}}+\frac{a\,b\,\left(a^2-ab+b^2\right)}{n^2}\,\frac{\mu_{\tau_{j-1,n}}}{\sigma^2_{\tau_{j-1,n}}}+O_p\left(\frac{1}{n^{5/2}}\right)\right)\\ &\quad -n^{1/2}\,(a-b)\,\sigma^2_{\tau_{j-1,n}}\left(\frac{ab}{n\,\sigma^2_{\tau_{j-1,n}}}+\frac{\mu_{\tau_{j-1,n}}a\,b\,(a-b)}{3n^{3/2}\,\sigma^4_{\tau_{j-1,n}}}+O_p\left(\frac{1}{n^2}\right)\right)\\ &= a\,b\,\left(2\,a^2-a\,b+2\,b^2\right)\,\frac{\mu_{\tau_{j-1,n}}}{3\,n\,\sigma^2_{\tau_{j-1,n}}}+O_p\left(\frac{1}{n^{3/2}}\right). \qquad (76)\end{aligned}$$

Hence,

$$\sum_{\tau_{j,n}\le t}\mathbb{E}_{\tau_{j-1,n}}\left[\xi_{j,n}\right]=a\,b\,\frac{2\,a^2-a\,b+2\,b^2}{3}\,\frac{1}{n}\sum_{\tau_{j,n}\le t}\frac{\mu_{\tau_{j-1,n}}}{\sigma^2_{\tau_{j-1,n}}}+O_p\left(n^{-1/2}\right)\xrightarrow{p}\frac{2\,a^2-a\,b+2\,b^2}{3}\int_0^t\mu_s\,ds,$$

where we used Lemma 9. In order to study the contribution coming from $\{\xi'_{j,n}\}$, we apply Itô's lemma to $\sigma^2_s-\sigma^2_{\tau_{j-1,n}}$, obtaining

$$\begin{aligned}&-\mathbb{E}_{\tau_{j-1,n}}\left[\xi'_{j,n}\right]\\ &=n^{1/2}\,(a-b)\,\mathbb{E}_{\tau_{j-1,n}}\left[\int_{\tau_{j-1,n}}^{\tau_{j,n}}\left(\int_{\tau_{j-1,n}}^{s}\left(2\,\sigma_u\,\mu'_u+\sigma'^2_u+\nu^2_u\right)\,du\right)ds\right]\\ &+n^{1/2}\,(a-b)\left(\mathbb{E}_{\tau_{j-1,n}}\left[\int_{\tau_{j-1,n}}^{\tau_{j,n}}\left(\int_{\tau_{j-1,n}}^{s}2\,\sigma_u\,\sigma'_u\,dW_u\right)ds\right]+\mathbb{E}_{\tau_{j-1,n}}\left[\int_{\tau_{j-1,n}}^{\tau_{j,n}}\left(\int_{\tau_{j-1,n}}^{s}2\,\sigma_u\,\nu_u\,dV_u\right)ds\right]\right)\\ &=2\,n^{1/2}\,(a-b)\,\sigma_{\tau_{j-1,n}}\,\sigma'_{\tau_{j-1,n}}\,\mathbb{E}_{\tau_{j-1,n}}\left[\int_{\tau_{j-1,n}}^{\tau_{j,n}}\left(W_s-W_{\tau_{j-1,n}}\right)ds\right]\\ &+2\,n^{1/2}\,(a-b)\,\sigma_{\tau_{j-1,n}}\,\nu_{\tau_{j-1,n}}\,\mathbb{E}_{\tau_{j-1,n}}\left[\int_{\tau_{j-1,n}}^{\tau_{j,n}}\left(V_s-V_{\tau_{j-1,n}}\right)ds\right]+O_p\left(n^{-3/2}\right)\\ &\text{(Eq. (44))}\\ &=2\,n^{1/2}\,(a-b)\,\sigma_{\tau_{j-1,n}}\,\sigma'_{\tau_{j-1,n}}\,\mathbb{E}_{\tau_{j-1,n}}\left[\Delta_{j,n}\,\Delta^n_j W\right]\\ &+2\,n^{1/2}\,(a-b)\,\sigma_{\tau_{j-1,n}}\,\nu_{\tau_{j-1,n}}\,\mathbb{E}_{\tau_{j-1,n}}\left[\Delta_{j,n}\,\Delta^n_j V\right]+O_p\left(n^{-3/2}\right)\\ &\text{(Lemma 2)}\\ &=2\,n^{1/2}\,(a-b)\,\sigma_{\tau_{j-1,n}}\,\sigma'_{\tau_{j-1,n}}\,\mathbb{E}_{\tau_{j-1,n}}\left[\Delta_{j,n}\,\Delta^n_j W\right]\\ &+2\,n^{1/2}\,(a-b)\,\sigma_{\tau_{j-1,n}}\,\nu_{\tau_{j-1,n}}\,\mathbb{E}_{\tau_{j-1,n}}\left[\int_{\tau_{j-1,n}}^{\tau_{j,n}}\Phi^{(V)}_{j,n}(s)\,ds\right]+O_p\left(n^{-3/2}\right)\\ &\text{(Eq. (71))}\\ &=2\,n^{1/2}\,(a-b)\,\sigma'_{\tau_{j-1,n}}\,\frac{ab(a-b)}{3\,n^{3/2}\,\sigma^2_{\tau_{j-1,n}}}+O_p\left(n^{-3/2}\right).\end{aligned}$$

In summary, using Lemma 9, we get

$$\sum_{\tau_{j,n}\le t}\mathbb{E}_{\tau_{j-1,n}}\left[\xi_{j,n}+\xi'_{j,n}\right]\xrightarrow{p}\frac{2\,a^2-a\,b+2\,b^2}{3}\int_0^t\mu_s\,ds-\frac{2}{3}\,(a-b)^2\int_0^t\sigma'_s\,ds.$$

For the second moment, we first notice that

$$\xi'^2_{j,n}=O_p\left(n^{-2}\right),\quad \xi_{j,n}\,\xi'_{j,n}=O_p\left(n^{-3/2}\right),$$

whence the only contribution derives from $\{\xi_{j,n}\}$. Hence, consider

$$\begin{aligned}
&\mathbb{E}_{\tau_{j-1,n}}\left[\xi_{j,n}^2\right] \\
&= n\,\mathbb{E}_{\tau_{j-1,n}}\left[n\left(\Delta_j^n X\right)^6 + (a-b)^2\,\sigma_{\tau_{j-1,n}}^4\,\Delta_{j,n}^2 - 2\,n^{1/2}\,(a-b)\,\sigma_{\tau_{j-1,n}}^2\,\Delta_{j,n}\left(\Delta_j^n X\right)^3\right] \\
&= n\left(\frac{ab\left(a^5+b^5\right)}{n^2\,(a+b)} + (a-b)^2\,\frac{a\,b}{3\,n^2}\left(a^2+3\,a\,b+b^2\right) - 2\,(a-b)\,\frac{ab(a-b)(a+b)^2}{3n^2} + O_p\left(n^{-5/2}\right)\right) \\
&= \frac{a\,b}{n}\left(\frac{\left(a^5+b^5\right)}{(a+b)} + \frac{1}{3}\,(a-b)^2\left(a^2+3\,a\,b+b^2\right) - \frac{2}{3}(a-b)^2(a+b)^2\right) + O_p\left(n^{-3/2}\right) \\
&= \frac{a\,b}{n}\left(a^2\,b^2 + \frac{2}{3}\,(a-b)^2\left(a^2+a\,b+b^2\right)\right) + O_p\left(n^{-3/2}\right).
\end{aligned} \tag{77}$$

Since $\sum_{\tau_{j,n}\le t}\mathbb{E}_{\tau_{j-1,n}}[\xi_{j,n}]^2 \xrightarrow{p} 0$, we get

$$\begin{aligned}
\sum_{\tau_{j,n}\le t}\left(\mathbb{E}_{\tau_{j-1,n}}\left[\xi_{j,n}^2\right] - \mathbb{E}_{\tau_{j-1,n}}\left[\xi_{j,n}\right]^2\right) &= \frac{N_{t,n}}{n}\,a\,b\left(a^2\,b^2 + \frac{2}{3}\,(a-b)^2\left(a^2+a\,b+b^2\right)\right) + o_p(1) \\
&\xrightarrow{p} \left((a\,b)^2 + \frac{2}{3}\,(a-b)^2\left(a^2+a\,b+b^2\right)\right)\int_0^t \sigma_s^2\,ds = \int_0^t\left(q_s^2+u_s^2\right)ds,
\end{aligned} \tag{78}$$

where the last expression, together with Eq. (81) below, is definitional for $\{q_s\}$ and $\{u_s\}$.

Consider, now, the limit of $\sum_{\tau_{j,n}\le t}\mathbb{E}_{\tau_{j-1,n}}[\xi_{j,n}\,\Delta_j^n W]$. Writing $\Delta_j^n X = \int_{\tau_{j-1,n}}^{\tau_{j,n}}\mu_s\,ds + \int_{\tau_{j-1,n}}^{\tau_{j,n}}\sigma_s\,dW_s$, we have a decomposition into five addends $\mathsf{A}_{n,t}(1),\ldots,\mathsf{A}_{n,t}(5)$, as follows:

$$\sum_{\tau_{j,n}\le t}\mathbb{E}_{\tau_{j-1,n}}\left[\xi_{j,n}\,\Delta_j^n W\right] = \sqrt{n}\sum_{\tau_{j,n}\le t}\mathbb{E}_{\tau_{j-1,n}}\left[\sqrt{n}\left(\Delta_j^n X\right)^3\Delta_j^n W - (a-b)\,\sigma_{\tau_{j-1,n}}^2\,\Delta_{j,n}\,\Delta_j^n W\right] = \sum_{q=1}^5 \mathsf{A}_{n,t}(q).$$

$\mathsf{A}_{n,t}(1)$ satisfies

$$\mathsf{A}_{n,t}(1) \doteq n\sum_{\tau_{j,n}\le t}\mathbb{E}_{\tau_{j-1,n}}\left[\left(\int_{\tau_{j-1,n}}^{\tau_{j,n}}\mu_s\,ds\right)^3\Delta_j^n W\right] = O_p\left(n^{-3/2}\right)\xrightarrow{p} 0.$$

$\mathsf{A}_{n,t}(2)$ satisfies

$$\begin{aligned}
\mathsf{A}_{n,t}(2) &\doteq n\sum_{\tau_{j,n}\le t}\mathbb{E}_{\tau_{j-1,n}}\left[\left(\int_{\tau_{j-1,n}}^{\tau_{j,n}}\sigma_s\,dW_s\right)^3\Delta_j^n W\right] = n\sum_{\tau_{j,n}\le t}\sigma_{\tau_{j-1,n}}^3\,\mathbb{E}_{\tau_{j-1,n}}\left[\left(\Delta_j^n W\right)^4\right] + O_p\left(n^{-3/2}\right) \\
&= \frac{ab\left(a^2-ab+b^2\right)}{n}\sum_{\tau_{j,n}\le t}\frac{1}{\sigma_{\tau_{j-1,n}}} + O_p\left(n^{-3/2}\right)\xrightarrow{p}\left(a^2-ab+b^2\right)\int_0^t\sigma_s\,ds,
\end{aligned} \tag{79}$$

given $\Delta_j^n X = \sigma_{\tau_{j-1,n}}\,\Delta_j^n W + O_p(n^{-1})$, the fourth expansion in Corollary A.1, i.e.,

$$\mathbb{E}_{\tau_{j-1,n}}\left[\left(\Delta_j^n W\right)^4\right] = \frac{ab\left(a^2-ab+b^2\right)}{\sigma_{\tau_{j-1,n}}^4\,n^2} + O_p\left(n^{-5/2}\right).$$

and Lemma 9. The following two A-addends are given by

$$\mathsf{A}_{n,t}(3) = 3\,n\sum_{\tau_{j,n}\le t}\mathbb{E}_{\tau_{j-1,n}}\left[\left(\int_{\tau_{j-1,n}}^{\tau_{j,n}}\mu_s\,ds\right)^2\left(\int_{\tau_{j-1,n}}^{\tau_{j,n}}\sigma_s\,dW_s\right)\Delta_j^n W\right] = O_p\left(n^{-1}\right)\xrightarrow{p} 0$$

and

$$\mathsf{A}_{n,t}(4) = 3\,n\sum_{\tau_{j,n}\le t}\mathbb{E}_{\tau_{j-1,n}}\left[\left(\int_{\tau_{j-1,n}}^{\tau_{j,n}}\mu_s\,ds\right)\left(\int_{\tau_{j-1,n}}^{\tau_{j,n}}\sigma_s\,dW_s\right)^2\Delta_j^n W\right] = O_p\left(n^{-1/2}\right)\xrightarrow{p} 0.$$

As for the last one, using Eq. (71), we have

$$\begin{aligned}\mathsf{A}_{n,t}(5) &= -\sqrt{n}\,(a-b)\sum_{\tau_{j,n}\le t}\sigma^2_{\tau_{j-1,n}}\,\mathbb{E}_{\tau_{j-1,n}}\left[\Delta_{j,n}\,\Delta_j^n W\right] = -\sqrt{n}\sum_{\tau_{j,n}\le t}\left(\frac{ab(a-b)^2}{3\,n^{3/2}\,\sigma_{\tau_{j-1,n}}}+O_p\left(n^{-2}\right)\right)\\ &= -\frac{a\,b\,(a-b)^2}{3\,n}\sum_{\tau_{j,n}\le t}\frac{1}{\sigma_{\tau_{j-1,n}}}+O_p\left(n^{-1/2}\right)\xrightarrow{p}-\frac{(a-b)^2}{3}\int_0^t\sigma_s\,ds. \qquad (80)\end{aligned}$$

Jointly, we obtain

$$\sum_{\tau_{j,n}\le t}\mathbb{E}_{\tau_{j-1,n}}\left[\xi_{j,n}\,\Delta_j^n W\right]\xrightarrow{p}\frac{1}{3}\left(2\,a^2-a\,b+2\,b^2\right)\int_0^t\sigma_s\,ds=\int_0^t q_s\,ds, \qquad (81)$$

where the last equation is definitional for $\{q_s\}$. In particular, $q_s = d_0(a,b)\,\sigma_s$. Eq. (78) and Eq. (81) give

$$u_s^2=\frac{2}{9}(a-b)^2\left(a^2+ab+b^2\right)\sigma_s^2=d_1(a,b)^2\,\sigma_s^2,$$

where $d_1(a,b)$ is defined as in Eq. (18) in the statement. Tedious computations show that

$$\sum_{\tau_{j,n}\le t}\mathbb{E}_{\tau_{j-1,n}}\left[\xi_{j,n}^4\right]\xrightarrow{p}0.$$

Let N be a bounded $\mathcal{F}_t$-martingale such that $[W,N]\equiv 0$. Consider the decomposition

$$\begin{aligned}\sum_{\tau_{j,n}\le t}\mathbb{E}_{\tau_{j-1,n}}\left[\xi_{j,n}\,\Delta_j^n N\right] &= n^{1/2}\sum_{\tau_{j,n}\le t}\mathbb{E}_{\tau_{j-1,n}}\left[n^{1/2}\left(X_{\tau_{j,n}}-X_{\tau_{j-1,n}}\right)^3\Delta_j^n N-(a-b)\,\sigma^2_{\tau_{j-1,n}}\,\Delta_{j,n}\,\Delta_j^n N\right]\\ &= \mathsf{B}_{n,t}(1)+\mathsf{B}_{n,t}(2),\end{aligned}$$

where $\mathsf{B}_{n,t}(1)\doteq n\sum_{\tau_{j,n}\le t}\mathbb{E}_{\tau_{j-1,n}}[(X_{\tau_{j,n}}-X_{\tau_{j-1,n}})^3\,\Delta_j^n N]$ and $\mathsf{B}_{n,t}(2)$ is defined accordingly.

By Lemma 2, we have $\mathsf{B}_{n,t}(2)\xrightarrow{p}0$. As for $\mathsf{B}_{n,t}(1)$, decompose further as

$$\begin{aligned}&\mathsf{B}_{n,t}(1)\\ &= n\sum_{\tau_{j,n}\le t}\mathbb{E}_{\tau_{j-1,n}}\left[\left(\int_{\tau_{j-1,n}}^{\tau_{j,n}}\mu_s\,ds\right)^3\Delta_j^n N\right]+n\sum_{\tau_{j,n}\le t}\mathbb{E}_{\tau_{j-1,n}}\left[\left(\int_{\tau_{j-1,n}}^{\tau_{j,n}}\sigma_s\,dW_s\right)^3\Delta_j^n N\right]\\ &+3\,n\sum_{\tau_{j,n}\le t}\mathbb{E}_{\tau_{j-1,n}}\left[\left(\int_{\tau_{j-1,n}}^{\tau_{j,n}}\mu_s\,ds\right)^2\left(\int_{\tau_{j-1,n}}^{\tau_{j,n}}\sigma_s\,dW_s\right)\Delta_j^n N\right]+3\,n\sum_{\tau_{j,n}\le t}\mathbb{E}_{\tau_{j-1,n}}\left[\left(\int_{\tau_{j-1,n}}^{\tau_{j,n}}\mu_s\,ds\right)\left(\int_{\tau_{j-1,n}}^{\tau_{j,n}}\sigma_s\,dW_s\right)^2\Delta_j^n N\right]\\ &= \mathsf{B}'_{n,t}(1)+\mathsf{B}'_{n,t}(2)+\mathsf{B}'_{n,t}(3)+\mathsf{B}'_{n,t}(4).\end{aligned}$$

Since $\int_{\tau_{j-1,n}}^{\tau_{j,n}}\mu_s\,ds=O_p(n^{-1})$ and N is continuous (being $\mathcal{F}_t$-adapted), we obtain $\mathsf{B}'_{n,t}(1)=o_p(n^{-1})$, $\mathsf{B}'_{n,t}(3)=o_p(n^{-1/2})$ and $\mathsf{B}'_{n,t}(4)=o_p(1)$. Regarding $\mathsf{B}'_{n,t}(2)$, using a martingale representation for $(\int_{\tau_{j-1,n}}^{\tau_{j,n}}\sigma_s\,dW_s)^3$, we obtain

$$\begin{aligned}\mathsf{B}'_{n,t}(2) &= n\sum_{\tau_{j,n}\le t}\mathbb{E}_{\tau_{j-1,n}}\left[\left(\mathbb{E}_{\tau_{j-1,n}}\left[\left(\int_{\tau_{j-1,n}}^{\tau_{j,n}}\sigma_s\,dW_s\right)^3\right]+\int_{\tau_{j-1,n}}^{\tau_{j,n}}\varphi_s\,dW_s\right)\int_{\tau_{j-1,n}}^{\tau_{j,n}}dN_s\right]\\ &= n\sum_{\tau_{j,n}\le t}\mathbb{E}_{\tau_{j-1,n}}\left[\int_{\tau_{j-1,n}}^{\tau_{j,n}}\varphi_s\,dW_s\int_{\tau_{j-1,n}}^{\tau_{j,n}}dN_s\right]=n\sum_{\tau_{j,n}\le t}\mathbb{E}_{\tau_{j-1,n}}\left[\int_{\tau_{j-1,n}}^{\tau_{j,n}}\varphi_s\,d[W,N]_s\right]=0.\end{aligned}$$

Summing up, $\mathsf{B}_{n,t}(1)\xrightarrow{p}0$ and $\sum_{\tau_{j,n}\le t}\mathbb{E}_{\tau_{j-1,n}}[\xi_{j,n}\,\Delta_j^n N]\xrightarrow{p}0$. □

Proof of Lemma 3. Because $\sigma_t>0$ is continuous, IV_t is continuous too and strictly increasing. Thus,

$$\mathrm{IV}_{\tau_{j,n}}-\mathrm{IV}_{\tau_{j-1,n}}=\frac{a}{n}.$$

For fixed j, on $[\tau_{j-1,n}, \tau_{j,n}]$, define the increment process $\delta_j^n \mathrm{IV}(t) \doteq \mathrm{IV}_t - \mathrm{IV}_{\tau_{j-1,n}}$. Thus, $\delta_j^n \mathrm{IV}(\tau_{j-1,n}) = 0$ and $\delta_j^n \mathrm{IV}(\tau_{j,n}) = a/n$. Consider, for all $n \in \mathbb{N}$, the function

$$u^{(n)}(y,\sigma) \doteq \frac{1}{\sigma^2}\left(\frac{a}{n} - y\right),$$

which satisfies the boundary conditions

$$\begin{cases} u^{(n)}\left(\delta_j^n \mathrm{IV}\left(\tau_{j,n}\right), \sigma_{\tau_{j,n}}\right) = u^{(n)}\left(a/n, \sigma_{\tau_{j,n}}\right) = 0, \\ u^{(n)}\left(\delta_j^n \mathrm{IV}\left(\tau_{j-1,n}\right), \sigma_{\tau_{j-1,n}}\right) = u^{(n)}\left(0, \sigma_{\tau_{j-1,n}}\right) = a/\left(n\,\sigma^2_{\tau_{j-1,n}}\right). \end{cases} \tag{82}$$

Itô's lemma applied to $U_t^{(n)} \doteq u^{(n)}(\delta_j^n \mathrm{IV}(t), \sigma_t)$ gives

$$\begin{aligned} & dU_t^{(n)} \\ & = u_y^{(n)}\, d\left(\delta_j^n \mathrm{IV}(t)\right) + u_\sigma^{(n)}\, d\sigma_t + \frac{1}{2} u_{\sigma\sigma}^{(n)}\, d[\sigma]_t \\ & = u_y^{(n)}\, \sigma_t^2\, dt + u_\sigma^{(n)}\left(\mu_t'\, dt + \sigma_t'\, dW_t + \nu_t\, dV_t\right) + \frac{1}{2} u_{\sigma\sigma}^{(n)}\left(\sigma_t'^2 + \nu_t^2\right) dt, \end{aligned}$$

where we have used the fact that the process $\delta_j^n \mathrm{IV}$ has finite variation. The derivatives are

$$\begin{cases} u_y^{(n)}(y,\sigma) = -1/\sigma^2, \\ u_\sigma^{(n)}(y,\sigma) = -2\left((a/n) - y\right)/\sigma^3, \\ u_{\sigma\sigma}^{(n)}(y,\sigma) = 6\left((a/n) - y\right)/\sigma^4, \end{cases}$$

yielding the following equation for the process $\{U^{(n)}\}$:

$$\begin{aligned} dU_t^{(n)} = & -dt + u_\sigma^{(n)}\left(\delta_j^n \mathrm{IV}(t), \sigma_t\right) \sigma_t'\, dW_t + u_\sigma^{(n)}\left(\delta_j^n \mathrm{IV}(t), \sigma_t\right) \nu_t\, dV_t \\ & + \left(u_\sigma^{(n)}\left(\delta_j^n \mathrm{IV}(t), \sigma_t\right) \mu_t' + \frac{1}{2}\, u_{\sigma\sigma}^{(n)}\left(\delta_j^n \mathrm{IV}(t), \sigma_t\right)\left(\sigma_t'^2 + \nu_t^2\right)\right) dt. \end{aligned} \tag{83}$$

Integrating Eq. (83) from $\tau_{j-1,n}$ to $\tau_{j,n}$, and using the boundary conditions in Eq. (82), we obtain the representation

$$\tau_{j,n} - \tau_{j-1,n} = \frac{a}{n\sigma^2_{\tau_{j-1,n}}} + \int_{\tau_{j-1,n}}^{\tau_{j,n}} \Phi_{j,n}^{(W)}(t)\, dW_t + \int_{\tau_{j-1,n}}^{\tau_{j,n}} \Phi_{j,n}^{(V)}(t)\, dV_t + \int_{\tau_{j-1,n}}^{\tau_{j,n}} \Phi_{j,n}^{(\Delta)}(t)\, dt, \tag{84}$$

where the coefficients have the following orders of probability

$$\begin{aligned} \Phi_{j,n}^{(W)}(t) &\doteq -2\left(\frac{a}{n} - \delta_j^n \mathrm{IV}(t)\right)\frac{\sigma_t'}{\sigma_t^3} = O_p\left(n^{-1}\right), \\ \Phi_{j,n}^{(V)}(t) &= -2\left(\frac{a}{n} - \delta_j^n \mathrm{IV}(t)\right)\frac{\nu_t}{\sigma_t^3} = O_p\left(n^{-1}\right), \\ \Phi_{j,n}^{(\Delta)}(t) &= -2\left(\frac{a}{n} - \delta_j^n \mathrm{IV}(t)\right)\frac{\mu_t'}{\sigma_t^3} + 3\left(\frac{a}{n} - \delta_j^n \mathrm{IV}(t)\right)\frac{\sigma_t'^2 + \nu_t^2}{\sigma_t^4} = O_p\left(n^{-1}\right). \end{aligned}$$

We, thus, have the expansion

$$\tau_{j,n} - \tau_{j-1,n} = \frac{a}{n\sigma^2_{\tau_{j-1,n}}} + O_p\left(n^{-3/2}\right).$$

□

Lemma 10. *Let Assumptions 1, 2 and 5 hold. Then, for any bounded càdlàg process G, we obtain*

$$\frac{1}{n}\sum_{\tau_{j,n} \le t} G_{\tau_{j-1,n}} \xrightarrow{p} \frac{1}{a}\int_0^t G_s\, \sigma_s^2\, ds. \tag{85}$$

In particular,

$$\frac{1}{n}\sum_{\tau_{j,n}\le t} 1 = \frac{N_{t,n}}{n} \xrightarrow{p} \frac{1}{a}\int_0^t \sigma_s^2\, ds.$$

Proof. The proof is just an application of the more general Lemma 7. □

Theorem A.2. *Let Assumptions 1, 2 and 5 hold. As $n \to \infty$, it holds*

$$\sum_{\tau_{j,n}\le t} \left(\Delta_j^n X\right)^2 \xrightarrow{p} \int_0^t \sigma_s^2\, ds \tag{86}$$

and, stably,

$$\sqrt{n}\left(\sum_{\tau_{j,n}\le t} \left(\Delta_j^n X\right)^2 - \int_0^t \sigma_s^2\, ds\right) \xrightarrow{\mathcal{L}-s} \sqrt{2\,a}\int_0^t \sigma_s\, dW'_s, \tag{87}$$

where W' is a Brownian motion independent of W. Moreover,

$$\sum_{\tau_{j,n}\le t} \sqrt{n}\left(\Delta_j^n X\right)^3 \xrightarrow{p} 0 \tag{88}$$

and, stably,

$$\sqrt{n}\sum_{\tau_{j,n}\le t} \sqrt{n}\left(\Delta_j^n X\right)^3 \xrightarrow{\mathcal{L}-s} 3\,a\int_0^t \mu_s\, ds + 3\,a\int_0^t \sigma_s\, dW_s + \sqrt{6}\,a\int_0^t \sigma_s\, dW'_s, \tag{89}$$

where, as above, W' is a Brownian motion independent of W.

Proof of Theorems 3.3 and A.2. The convergence in Eq. (86) only requires $\sup_j \Delta_{j,n} \xrightarrow{p} 0$, which is trivially true. In what follows, we will repeatedly use the following approximation. Recall that the filtration $\mathcal{F}_t$ is assumed to be generated by a finite number of Brownian motions. Thus, for any $\mathcal{F}_t$-adapted càdlàg bounded process $\{\alpha\}$, we have

$$\alpha_t = \alpha_{\tau_{j-1,n}} + \alpha_t - \alpha_{\tau_{j-1,n}} = \alpha_{\tau_{j-1,n}} + O_p\left(n^{-1/2}\right),$$

uniformly on $[\tau_{j-1,n}, \tau_{j,n}]$. Now, write

$$\begin{aligned}\sqrt{n}\left(\sum_{\tau_{j,n}\le t} \left(\Delta_j^n X\right)^2 - \int_0^t \sigma_s^2\, ds\right) &= \sqrt{n}\sum_{\tau_{j,n}\le t}\left(\left(\Delta_j^n X\right)^2 - \int_{\tau_{j-1,n}}^{\tau_{j,n}} \sigma_s^2\, ds\right) - \sqrt{n}\int_{\tau_{N_{t,n},n}}^t \sigma_s^2\, ds = \\ &= \sqrt{n}\sum_{\tau_{j,n}\le t}\left(\left(\Delta_j^n X\right)^2 - \frac{a}{n}\right) + o_p(1) = \sum_{\tau_{j,n}\le t} \xi_{j,n} + o_p(1)\end{aligned} \tag{90}$$

with $\xi_{j,n} \doteq \sqrt{n}\,((\Delta_j^n X)^2 - a/n)$, where we have used Assumption 5 (which implies $\int_{\tau_{j-1,n}}^{\tau_{j,n}} \sigma_s^2\, ds = a/n$) and the fact that

$$\sqrt{n}\int_{\tau_{N_{t,n},n}}^t \sigma_s^2\, ds \xrightarrow{p} 0.$$

Define $A_t \doteq \int_0^t \mu_s\, ds$ and $M_t \doteq \int_0^t \sigma_s\, dW_s$ to obtain $\Delta_j^n X = \Delta_j^n A + \Delta_j^n M$, where $\Delta_j^n A = \int_{\tau_{j-1,n}}^{\tau_{j,n}} \mu_s\, ds$ and $\Delta_j^n M = \int_{\tau_{j-1,n}}^{\tau_{j,n}} \sigma_s\, dW_s$. We have the following approximation:

$$\Delta_j^n A = \mu_{\tau_{j-1,n}}\,\Delta_{j,n} + O_p\left(n^{-3/2}\right) = \frac{a\,\mu_{\tau_{j-1,n}}}{n\,\sigma^2_{\tau_{j-1,n}}} + O_p\left(n^{-3/2}\right),$$

where we have used the representation in Eq. (84) for $\Delta_{j,n}$. Since, trivially $(\Delta_j^n A)^2 = O_p(n^{-2})$, we obtain

$$\left(\Delta_j^n X\right)^2 = \left(\Delta_j^n M\right)^2 + 2\,\Delta_j^n A\,\Delta_j^n M + O_p\left(n^{-2}\right) = \left(\Delta_j^n M\right)^2 + \frac{2\,a\,\mu_{\tau_{j-1,n}}}{n\,\sigma^2_{\tau_{j-1,n}}}\,\Delta_j^n M + O_p\left(n^{-2}\right). \tag{91}$$

Because $\mathbb{E}_{\tau_{j-1,n}}[\Delta_j^n M] = 0$, we have

$$\begin{aligned}\sum_{\tau_{j,n}\leq t}\mathbb{E}_{\tau_{j-1,n}}\left[\xi_{j,n}\right] &= \sqrt{n}\sum_{\tau_{j,n}\leq t}\left(\mathbb{E}_{\tau_{j-1,n}}\left[\left(\Delta_j^n M\right)^2\right]-\frac{a}{n}+O_p\left(n^{-2}\right)\right)\\ &= \sqrt{n}\sum_{\tau_{j,n}\leq t}\left(\mathbb{E}_{\tau_{j-1,n}}\left[\int_{\tau_{j-1,n}}^{\tau_{j,n}}\sigma_s^2\,ds\right]-\frac{a}{n}\right)+O_p\left(n^{-1/2}\right)=O_p\left(n^{-1/2}\right)\xrightarrow{p}0,\end{aligned}\tag{92}$$

where we have used (again) Assumption 5 and the second Bartlett identity (Itô isometry). Now, consider

$$\begin{aligned}\xi_{j,n}^2 &= n\left(\left(\Delta_j^n X\right)^4+\frac{a^2}{n^2}-\frac{2\,a}{n}\left(\Delta_j^n X\right)^2\right)\\ &= n\left(\left(\Delta_j^n M\right)^4+\frac{a^2}{n^2}-\frac{2\,a}{n}\left(\Delta_j^n X\right)^2\right)+O_p\left(n^{-3/2}\right)\\ &= n\left(\left(\Delta_j^n M\right)^4+\frac{a^2}{n^2}-\frac{2\,a}{n}\left(\Delta_j^n M\right)^2-\frac{4\,a^2\,\mu_{\tau_{j-1,n}}}{n^2\,\sigma^2_{\tau_{j-1,n}}}\Delta_j^n M\right)+O_p\left(n^{-3/2}\right),\end{aligned}\tag{93}$$

where, in the first equation, we have used Eq. (91) and the expansion

$$\left(\Delta_j^n X\right)^4=\left(\Delta_j^n M\right)^4+O_p\left(n^{-5/2}\right).$$

We, thus, have

$$\begin{aligned}\mathbb{E}_{\tau_{j-1,n}}\left[\xi_{j,n}^2\right] &= n\left(\mathbb{E}_{\tau_{j-1,n}}\left[\left(\Delta_j^n M\right)^4\right]-\frac{a^2}{n^2}\right)+O_p\left(n^{-3/2}\right)\\ &= n\left(6\,\mathbb{E}_{\tau_{j-1,n}}\left[\left(\int_{\tau_{j-1,n}}^{\tau_{j,n}}\sigma_s^2\,ds\right)\left(\Delta_j^n M\right)^2\right]-3\,\mathbb{E}_{\tau_{j-1,n}}\left[\left(\int_{\tau_{j-1,n}}^{\tau_{j,n}}\sigma_s^2\,ds\right)^2\right]-\frac{a^2}{n^2}\right)+O_p\left(n^{-3/2}\right)\\ &= n\left(\frac{6\,a}{n}\,\mathbb{E}_{\tau_{j-1,n}}\left[\left(\Delta_j^n M\right)^2\right]-3\,\frac{a^2}{n^2}-\frac{a^2}{n^2}\right)+O_p\left(n^{-3/2}\right)\\ &= n\left(\frac{6\,a}{n}\,\mathbb{E}_{\tau_{j-1,n}}\left[\left(\Delta_j^n M\right)^2\right]-3\,\frac{a^2}{n^2}-\frac{a^2}{n^2}\right)+O_p\left(n^{-3/2}\right)=\frac{2\,a^2}{n}+O_p\left(n^{-3/2}\right),\end{aligned}\tag{94}$$

and

$$\sum_{\tau_{j,n}\leq t}\mathbb{E}_{\tau_{j-1,n}}\left[\xi_{j,n}^2\right]=\frac{2\,a^2}{n}\sum_{\tau_{j,n}\leq t}1+O_p\left(n^{-1/2}\right)\xrightarrow{p}2\,a\int_0^t\sigma_s^2\,ds.$$

We also have $\sum_{\tau_{j,n}\leq t}\mathbb{E}_{\tau_{j-1,n}}[\xi_{j,n}]^2=O_p(n^{-2})$. To finalize the calculation of the asymptotic variance, we compute

the limit of

$$
\begin{aligned}
&\sum_{\tau_{j,n}\le t} \mathbb{E}_{\tau_{j-1,n}}\left[\xi_{j,n}\,\Delta_j^n W\right] \\
&= \sqrt{n}\sum_{\tau_{j,n}\le t} \mathbb{E}_{\tau_{j-1,n}}\left[\left(\Delta_j^n X\right)^2 \Delta_j^n W\right] \\
&= \sqrt{n}\sum_{\tau_{j,n}\le t} \mathbb{E}_{\tau_{j-1,n}}\left[\left(\left(\Delta_j^n M\right)^2 + \frac{2\,a\,\mu_{\tau_{j-1,n}}}{n\,\sigma^2_{\tau_{j-1,n}}}\,\Delta_j^n M + O_p\left(n^{-2}\right)\right)\Delta_j^n W\right] \\
&= \sqrt{n}\sum_{\tau_{j,n}\le t} \mathbb{E}_{\tau_{j-1,n}}\left[\left(\left(\Delta_j^n M\right)^2 \Delta_j^n W + O_p\left(n^{-2}\right)\right)\right] \\
&= \sqrt{n}\sum_{\tau_{j,n}\le t} \sigma^2_{\tau_{j-1,n}}\,\mathbb{E}_{\tau_{j-1,n}}\left[\left(\left(\Delta_j^n W\right)^3 + O_p\left(n^{-2}\right)\right)\right] \\
&= \sqrt{n}\sum_{\tau_{j,n}\le t} \sigma^2_{\tau_{j-1,n}}\,\mathbb{E}_{\tau_{j-1,n}}\left[3\,\Delta_{j,n}\,\Delta_j^n W\right] + O_p\left(n^{-1/2}\right) \\
&= \sqrt{n}\sum_{\tau_{j,n}\le t} \sigma^2_{\tau_{j-1,n}}\,\mathbb{E}_{\tau_{j-1,n}}\left[3\left(\frac{a}{n\,\sigma^2_{\tau_{j-1,n}}} + \int_{\tau_{j-1,n}}^{\tau_{j,n}} \Phi^{(W)}_{j,n}(s)\,dW_s + \int_{\tau_{j-1,n}}^{\tau_{j,n}} \Phi^{(V)}_{j,n}(s)\,dV_s + \int_{\tau_{j-1,n}}^{\tau_{j,n}} \Phi^{(\Delta)}_{j,n}(s)\,ds\right)\Delta_j^n W\right] \\
&+ O_p\left(n^{-1/2}\right) \\
&= O_p\left(n^{-1/2}\right).
\end{aligned}
\tag{95}
$$

Turning to the tricity, consider the expansion

$$
\begin{aligned}
\mathbb{E}_{\tau_{j-1,n}}\left[\left(\Delta_j^n X\right)^3\right] &= \mathbb{E}_{\tau_{j-1,n}}\left[\left(\int_{\tau_{j-1,n}}^{\tau_{j,n}} \mu_s\,ds + \int_{\tau_{j-1,n}}^{\tau_{j,n}} \sigma_s\,dW_s\right)^3\right] = \mathbb{E}_{\tau_{j-1,n}}\left[\left(\int_{\tau_{j-1,n}}^{\tau_{j,n}} \mu_s\,ds\right)^3 + \left(\int_{\tau_{j-1,n}}^{\tau_{j,n}} \sigma_s\,dW_s\right)^3\right] + \\
&\quad + \mathbb{E}_{\tau_{j-1,n}}\left[3\left(\int_{\tau_{j-1,n}}^{\tau_{j,n}} \mu_s\,ds\right)^2\left(\int_{\tau_{j-1,n}}^{\tau_{j,n}} \sigma_s\,dW_s\right) + 3\left(\int_{\tau_{j-1,n}}^{\tau_{j,n}} \mu_s\,ds\right)\left(\int_{\tau_{j-1,n}}^{\tau_{j,n}} \sigma_s\,dW_s\right)^2\right] \\
&= \mathbb{E}_{\tau_{j-1,n}}\left[\left(\int_{\tau_{j-1,n}}^{\tau_{j,n}} \sigma_s\,dW_s\right)^3\right] + O_p\left(n^{-2}\right).
\end{aligned}
$$

We have the following chain of identities:

$$
\begin{aligned}
&\sum_{\tau_{j,n}\le t}\sqrt{n}\,\mathbb{E}_{\tau_{j-1,n}}\left[\left(\Delta_j^n X\right)^3\right]\\
&=\sum_{\tau_{j,n}\le t}\sqrt{n}\,\mathbb{E}_{\tau_{j-1,n}}\left[\left(\int_{\tau_{j-1,n}}^{\tau_{j,n}}\sigma_s\,dW_s\right)^3\right]+O_p\left(n^{-1/2}\right)\\
&\text{(Eq. 45)}\\
&=3\sum_{\tau_{j,n}\le t}\sqrt{n}\,\mathbb{E}_{\tau_{j-1,n}}\left[\int_{\tau_{j-1,n}}^{\tau_{j,n}}\sigma_s^2\left(\int_{\tau_{j-1,n}}^{s}\sigma_u\,dW_u\right)ds\right]+O_p\left(n^{-1/2}\right)\\
&\text{(Eq. (97) below)}\\
&=3\sum_{\tau_{j,n}\le t}\sqrt{n}\,\sigma^3_{\tau_{j-1,n}}\,\mathbb{E}_{\tau_{j-1,n}}\left[\int_{\tau_{j-1,n}}^{\tau_{j,n}}\left(\int_{\tau_{j-1,n}}^{s}dW_u\right)ds\right]+O_p\left(n^{-1/2}\right)\\
&\text{(Eq. 44)}\\
&=3\sum_{\tau_{j,n}\le t}\sqrt{n}\,\sigma^3_{\tau_{j-1,n}}\,\mathbb{E}_{\tau_{j-1,n}}\left[\left(\tau_{j,n}-\tau_{j-1,n}\right)\left(W_{\tau_{j,n}}-W_{\tau_{j-1,n}}\right)\right]+O_p\left(n^{-1/2}\right)\\
&=3\sum_{\tau_{j,n}\le t}\sqrt{n}\,\sigma^3_{\tau_{j-1,n}}\,\mathbb{E}_{\tau_{j-1,n}}\left[\left(\int_{\tau_{j-1,n}}^{\tau_{j,n}}\Phi^{(W)}_{j,n}(t)\,dW_t+\int_{\tau_{j-1,n}}^{\tau_{j,n}}\Phi^{(V)}_{j,n}(t)\,dV_t+\int_{\tau_{j-1,n}}^{\tau_{j,n}}\Phi^{(\Delta)}_{j,n}(t)\,dt\right)\Delta_j^n W\right]\\
&+O_p\left(n^{-1/2}\right)\\
&=O_p\left(n^{-1/2}\right),
\end{aligned}
\tag{96}
$$

where we have used Eq.(84) for $\Delta_{j,n}$, the fact that $\mathbb{E}_{\tau_{j-1,n}}[\frac{a}{n\,\sigma^2_{\tau_{j-1,n}}}\Delta_j^n W]=0$, and the expansion

$$
\begin{aligned}
&\mathbb{E}_{\tau_{j-1,n}}\left[\int_{\tau_{j-1,n}}^{\tau_{j,n}}\sigma_s^2\left(\int_{\tau_{j-1,n}}^{s}\sigma_u\,dW_u\right)ds\right]\\
&=\mathbb{E}_{\tau_{j-1,n}}\left[\int_{\tau_{j-1,n}}^{\tau_{j,n}}\left(\sigma_s^2-\sigma^2_{\tau_{j-1,n}}+\sigma^2_{\tau_{j-1,n}}\right)\left(\int_{\tau_{j-1,n}}^{s}\left(\sigma_u-\sigma_{\tau_{j-1,n}}+\sigma_{\tau_{j-1,n}}\right)dW_u\right)ds\right]\\
&=\underbrace{\mathbb{E}_{\tau_{j-1,n}}\left[\int_{\tau_{j-1,n}}^{\tau_{j,n}}\left(\sigma_s^2-\sigma^2_{\tau_{j-1,n}}\right)\left(\int_{\tau_{j-1,n}}^{s}\left(\sigma_u-\sigma_{\tau_{j-1,n}}\right)dW_u\right)ds\right]}_{O_p(n^{-5/2})}\\
&+\underbrace{\mathbb{E}_{\tau_{j-1,n}}\left[\int_{\tau_{j-1,n}}^{\tau_{j,n}}\left(\sigma_s^2-\sigma^2_{\tau_{j-1,n}}\right)\left(\int_{\tau_{j-1,n}}^{s}\sigma_{\tau_{j-1,n}}\,dW_u\right)ds\right]}_{O_p(n^{-2})}\\
&+\underbrace{\mathbb{E}_{\tau_{j-1,n}}\left[\int_{\tau_{j-1,n}}^{\tau_{j,n}}\sigma^2_{\tau_{j-1,n}}\left(\int_{\tau_{j-1,n}}^{s}\left(\sigma_u-\sigma_{\tau_{j-1,n}}\right)dW_u\right)ds\right]}_{O_p(n^{-2})}+\underbrace{\sigma^3_{\tau_{j-1,n}}\,\mathbb{E}_{\tau_{j-1,n}}\left[\int_{\tau_{j-1,n}}^{\tau_{j,n}}\left(\int_{\tau_{j-1,n}}^{s}dW_u\right)ds\right]}_{O_p(n^{-3/2})}.
\end{aligned}
\tag{97}
$$

To compute the limit of $\sqrt{n}\sum_{\tau_{j,n}\le t}\sqrt{n}\,\mathbb{E}_{\tau_{j-1,n}}[(\Delta_j^n X)^3]$, consider again the quantity $\mathbb{E}_{\tau_{j-1,n}}[(\Delta_j^n X)^3]$, which gives

$$\begin{aligned}
\mathbb{E}_{\tau_{j-1,n}}\left[\left(\Delta_j^n X\right)^3\right] &= \mathbb{E}_{\tau_{j-1,n}}\left[\left(\int_{\tau_{j-1,n}}^{\tau_{j,n}}\mu_s\,ds+\int_{\tau_{j-1,n}}^{\tau_{j,n}}\sigma_s\,dW_s\right)^3\right] = \mathbb{E}_{\tau_{j-1,n}}\left[\underbrace{\left(\int_{\tau_{j-1,n}}^{\tau_{j,n}}\mu_s\,ds\right)^3}_{O_p(n^{-3})}+\underbrace{\left(\int_{\tau_{j-1,n}}^{\tau_{j,n}}\sigma_s\,dW_s\right)^3}_{O_p(n^{-3/2})}\right]+\\
&\quad+\mathbb{E}_{\tau_{j-1,n}}\left[3\underbrace{\left(\int_{\tau_{j-1,n}}^{\tau_{j,n}}\mu_s\,ds\right)^2\left(\int_{\tau_{j-1,n}}^{\tau_{j,n}}\sigma_s\,dW_s\right)}_{O_p(n^{-5/2})}+3\underbrace{\left(\int_{\tau_{j-1,n}}^{\tau_{j,n}}\mu_s\,ds\right)\left(\int_{\tau_{j-1,n}}^{\tau_{j,n}}\sigma_s\,dW_s\right)^2}_{O_p(n^{-2})}\right]\\
&= \mathbb{E}_{\tau_{j-1,n}}\left[\left(\int_{\tau_{j-1,n}}^{\tau_{j,n}}\sigma_s\,dW_s\right)^3\right]+3\,\mathbb{E}_{\tau_{j-1,n}}\left[\left(\int_{\tau_{j-1,n}}^{\tau_{j,n}}\mu_s\,ds\right)\left(\int_{\tau_{j-1,n}}^{\tau_{j,n}}\sigma_s\,dW_s\right)^2\right]+O_p\left(n^{-5/2}\right). \quad (98)
\end{aligned}$$

We thus have

$$\begin{aligned}
&\sqrt{n}\sum_{\tau_{j,n}\le t}\sqrt{n}\,\mathbb{E}_{\tau_{j-1,n}}\left[\left(\Delta_j^n X\right)^3\right]=\\
&= \sum_{\tau_{j,n}\le t} n\,\mathbb{E}_{\tau_{j-1,n}}\left[\left(\int_{\tau_{j-1,n}}^{\tau_{j,n}}\sigma_s\,dW_s\right)^3\right]+3\,n\sum_{\tau_{j,n}\le t}\mathbb{E}_{\tau_{j-1,n}}\left[\left(\int_{\tau_{j-1,n}}^{\tau_{j,n}}\mu_s\,ds\right)\left(\int_{\tau_{j-1,n}}^{\tau_{j,n}}\sigma_s\,dW_s\right)^2\right]+O_p\left(n^{-1/2}\right)\\
&= 3\sum_{\tau_{j,n}\le t} n\,\mathbb{E}_{\tau_{j-1,n}}\left[\int_{\tau_{j-1,n}}^{\tau_{j,n}}\sigma_s^2\left(\int_{\tau_{j-1,n}}^{s}\sigma_u\,dW_u\right)ds\right]+3\,n\sum_{\tau_{j,n}\le t}\mathbb{E}_{\tau_{j-1,n}}\left[\int_{\tau_{j-1,n}}^{\tau_{j,n}}\left(\int_{\tau_{j-1,n}}^{s}\mu_u\,du\right)\sigma_s^2\,ds\right]\\
&\quad+3\,n\sum_{\tau_{j,n}\le t}\mathbb{E}_{\tau_{j-1,n}}\left[\int_{\tau_{j-1,n}}^{\tau_{j,n}}\mu_s\left(\int_{\tau_{j-1,n}}^{s}\sigma_u\,dW_u\right)^2ds\right]+O_p\left(n^{-1/2}\right)\doteq 3\left(\mathsf{S}_n(1)+\mathsf{S}_n(2)+\mathsf{S}_n(3)\right)+O_p\left(n^{-1/2}\right),
\end{aligned}$$

having used, in the second equation, Eq. (45) and integration by parts[15]. Now, define $\varpi_{j-1}(s)\doteq\sigma_s^2-\sigma^2_{\tau_{j-1,n}}$,

[15]Let $Z_t=\int_0^t\mu_s\,ds$ and $U_t=M_t^2$ with $M_t=\int_0^t\sigma_s\,dW_s$. Hence

$$dU_t=2M_t\,dM_t+d[M]_t=2M_t\,\sigma_t\,dW_t+\sigma_t^2\,dt.$$

Since $[Z,U]_t=0$ we obtain

$$d(Z_t\,U_t)=Z_t\,dU_t+U_t\,dZ_t+d[Z,U]_t=Z_t\left(2M_t\sigma_t\,dW_t+\sigma_t^2\,dt\right)+U_t\,\mu_t\,dt.$$

Integrating from 0 to t, and noticing that $Z_0=0$, $U_0=0$, we obtain

$$Z_t\,U_t=2\int_0^t Z_s\,M_s\,\sigma_s\,dW_s+\int_0^t Z_s\,\sigma_s^2\,ds+\int_0^t\mu_s\,U_s\,ds.$$

Since $U_s=M_s^2$, this can be written as

$$\left(\int_0^t\mu_s\,ds\right)\left(\int_0^t\sigma_s\,dW_s\right)^2=2\int_0^t\left(\int_0^s\mu_u\,du\right)\left(\int_0^s\sigma_u\,dW_u\right)\sigma_s\,dW_s+\int_0^t\left(\int_0^s\mu_u\,du\right)\sigma_s^2\,ds+\int_0^t\mu_s\left(\int_0^s\sigma_u\,dW_u\right)^2ds,$$

giving

$$\mathbb{E}_0\left[\left(\int_0^t\mu_s\,ds\right)\left(\int_0^t\sigma_s\,dW_s\right)^2\right]=\mathbb{E}_0\left[\int_0^t\left(\int_0^s\mu_u\,du\right)\sigma_s^2\,ds+\int_0^t\mu_s\left(\int_0^s\sigma_u\,dW_u\right)^2ds\right].$$

$\varrho_{j-1}(s) \doteq \sigma_s - \sigma_{\tau_{j-1,n}}$ and use Eq. (97), noticing that

$$
\begin{aligned}
&\mathsf{S}_n(1) \\
&= \sum_{\tau_{j,n}\le t} n\,\mathbb{E}_{\tau_{j-1,n}}\left[\int_{\tau_{j-1,n}}^{\tau_{j,n}} \sigma_s^2 \left(\int_{\tau_{j-1,n}}^{s} \sigma_u\, dW_u\right) ds\right] \\
&= \sum_{\tau_{j,n}\le t} n\,\mathbb{E}_{\tau_{j-1,n}}\left[\int_{\tau_{j-1,n}}^{\tau_{j,n}} \varpi_{j-1}(s) \left(\int_{\tau_{j-1,n}}^{s} \sigma_{\tau_{j-1,n}}\, dW_u\right) ds\right] \\
&+ \sum_{\tau_{j,n}\le t} n\,\mathbb{E}_{\tau_{j-1,n}}\left[\int_{\tau_{j-1,n}}^{\tau_{j,n}} \sigma^2_{\tau_{j-1,n}} \left(\int_{\tau_{j-1,n}}^{s} \varrho_{j-1}(u)\, dW_u\right) ds\right] + \sum_{\tau_{j,n}\le t} n\,\sigma^3_{\tau_{j-1,n}}\,\mathbb{E}_{\tau_{j-1,n}}\left[\int_{\tau_{j-1,n}}^{\tau_{j,n}} \left(\int_{\tau_{j-1,n}}^{s} dW_u\right) ds\right] \\
&+ O_p\left(n^{-1/2}\right) \\
&\doteq \mathsf{S}'_n(1) + \mathsf{S}'_n(2) + \mathsf{S}'_n(3) + O_p\left(n^{-1/2}\right).
\end{aligned}
\tag{99}
$$

Next, write

$$
\begin{aligned}
\mathsf{S}'_n(1) &= \sum_{\tau_{j,n}\le t} n\,\mathbb{E}_{\tau_{j-1,n}}\left[\int_{\tau_{j-1,n}}^{\tau_{j,n}} \left(\sigma_s^2 - \sigma^2_{\tau_{j-1,n}}\right) \left(\int_{\tau_{j-1,n}}^{s} \sigma_{\tau_{j-1,n}}\, dW_u\right) ds\right] \\
&= \sum_{\tau_{j,n}\le t} n\,\sigma_{\tau_{j-1,n}}\,\mathbb{E}_{\tau_{j-1,n}}\left[\int_{\tau_{j-1,n}}^{\tau_{j,n}} \left(\sigma_s^2 - \sigma^2_{\tau_{j-1,n}}\right) \left(\int_{\tau_{j-1,n}}^{s} dW_u\right) ds\right] \\
&= \sum_{\tau_{j,n}\le t} n\,\sigma_{\tau_{j-1,n}}\,\mathbb{E}_{\tau_{j-1,n}}\left[\int_{\tau_{j-1,n}}^{\tau_{j,n}} \left(\sigma_s^2 - \sigma^2_{\tau_{j-1,n}}\right) \left(W_s - W_{\tau_{j-1,n}}\right) ds\right].
\end{aligned}
$$

Consider the finite-variation process $A_s \doteq \int_{\tau_{j-1,n}}^{s} (\sigma_u^2 - \sigma^2_{\tau_{j-1,n}})\, du$. Using Assumption 5, write

$$
A_{\tau_{j,n}} = \int_{\tau_{j-1,n}}^{\tau_{j,n}} \sigma_u^2\, du - \sigma^2_{\tau_{j-1,n}}\,\Delta_{j,n} = \frac{a}{n} - \sigma^2_{\tau_{j-1,n}}\,\Delta_{j,n}. \tag{100}
$$

Since $[W, A]_t \equiv 0$, we obtain

$$
d\left(\left(W_s - W_{\tau_{j-1,n}}\right) A_s\right) = A_s\, dW_s + \left(W_s - W_{\tau_{j-1,n}}\right) dA_s = A_s\, dW_s + \left(W_s - W_{\tau_{j-1,n}}\right)\left(\sigma_s^2 - \sigma^2_{\tau_{j-1,n}}\right) ds.
$$

Integrating both sides yields

$$
\left(W_{\tau_{j,n}} - W_{\tau_{j-1,n}}\right) A_{\tau_{j,n}} = \int_{\tau_{j-1,n}}^{\tau_{j,n}} A_s\, dW_s + \int_{\tau_{j-1,n}}^{\tau_{j,n}} \left(W_s - W_{\tau_{j-1,n}}\right)\left(\sigma_s^2 - \sigma^2_{\tau_{j-1,n}}\right) ds.
$$

Thus, using the fact that $\int_{\tau_{j-1,n}}^{\cdot} A_s\, dW_s$ is a martingale, we have

$$
\begin{aligned}
\mathbb{E}_{\tau_{j-1,n}}\left[\int_{\tau_{j-1,n}}^{\tau_{j,n}} \left(W_s - W_{\tau_{j-1,n}}\right)\left(\sigma_s^2 - \sigma^2_{\tau_{j-1,n}}\right) ds\right] &= \mathbb{E}_{\tau_{j-1,n}}\left[\left(W_{\tau_{j,n}} - W_{\tau_{j-1,n}}\right) A_{\tau_{j,n}}\right] \\
&= -\sigma^2_{\tau_{j-1,n}}\,\mathbb{E}_{\tau_{j-1,n}}\left[\Delta_{j,n}\left(W_{\tau_{j,n}} - W_{\tau_{j-1,n}}\right)\right],
\end{aligned}
$$

because of Eq. (100). Thus, Eq. (84) now gives

$$
\begin{aligned}
&\mathbb{E}_{\tau_{j-1,n}}\left[\int_{\tau_{j-1,n}}^{\tau_{j,n}} \left(W_s - W_{\tau_{j-1,n}}\right)\left(\sigma_s^2 - \sigma^2_{\tau_{j-1,n}}\right) ds\right] \\
&= -\sigma^2_{\tau_{j-1,n}}\,\mathbb{E}_{\tau_{j-1,n}}\left[\left(\int_{\tau_{j-1,n}}^{\tau_{j,n}} \Phi^{(W)}_{j,n}(t)\, dW_t + \int_{\tau_{j-1,n}}^{\tau_{j,n}} \Phi^{(\Delta)}_{j,n}(t)\, dt\right)\left(W_{\tau_{j,n}} - W_{\tau_{j-1,n}}\right)\right] \\
&= -\sigma^2_{\tau_{j-1,n}}\,\mathbb{E}_{\tau_{j-1,n}}\left[\int_{\tau_{j-1,n}}^{\tau_{j,n}} \Phi^{(W)}_{j,n}(t)\, dt\right] + O_p\left(n^{-5/2}\right),
\end{aligned}
$$

which results in

$$\mathsf{S}'_n(1) = -\sum_{\tau_{j,n}\le t} n\,\sigma^3_{\tau_{j-1,n}}\,\mathbb{E}_{\tau_{j-1,n}}\left[\int_{\tau_{j-1,n}}^{\tau_{j,n}} \Phi^{(W)}_{j,n}(t)\,dt\right] + O_p\left(n^{-1/2}\right).$$

Replicating the same logic[16] also gives

$$\begin{aligned}
&\mathbb{E}_{\tau_{j-1,n}}\left[\int_{\tau_{j-1,n}}^{\tau_{j,n}} \sigma^2_{\tau_{j-1,n}}\left(\int_{\tau_{j-1,n}}^{s}\left(\sigma_u - \sigma_{\tau_{j-1,n}}\right)dW_u\right)ds\right]\\
&= \sigma^2_{\tau_{j-1,n}}\,\mathbb{E}_{\tau_{j-1,n}}\left[\left(\int_{\tau_{j-1,n}}^{\tau_{j,n}}\left(\sigma_s - \sigma_{\tau_{j-1,n}}\right)dW_s\right)\Delta_{j,n}\right]\\
&= \sigma^2_{\tau_{j-1,n}}\,\mathbb{E}_{\tau_{j-1,n}}\left[\left(\int_{\tau_{j-1,n}}^{\tau_{j,n}}\left(\sigma_s - \sigma_{\tau_{j-1,n}}\right)\Phi^{(W)}_{j,n}(s)\,ds\right)\right] + O_p\left(n^{-3}\right)\\
&= O_p\left(n^{-5/2}\right),
\end{aligned}$$

resulting in $\mathsf{S}'_n(2) = O_p(n^{-1/2})$. Regarding $\mathsf{S}'_n(3)$, we obtain

$$\begin{aligned}
\mathsf{S}'_n(3) &= \sum_{\tau_{j,n}\le t} n\,\sigma^3_{\tau_{j-1,n}}\,\mathbb{E}_{\tau_{j-1,n}}\left[\int_{\tau_{j-1,n}}^{\tau_{j,n}}\left(W_s - W_{\tau_{j-1,n}}\right)ds\right]\\
\text{(Eq. (44))} &= \sum_{\tau_{j,n}\le t} n\,\sigma^3_{\tau_{j-1,n}}\,\mathbb{E}_{\tau_{j-1,n}}\left[\left(\tau_{j,n} - \tau_{j-1,n}\right)\left(W_{\tau_{j,n}} - W_{\tau_{j-1,n}}\right)\right]\\
\text{(Eq. (84))} &= \sum_{\tau_{j,n}\le t} n\,\sigma^3_{\tau_{j-1,n}}\,\mathbb{E}_{\tau_{j-1,n}}\left[\int_{\tau_{j-1,n}}^{\tau_{j,n}}\Phi^{(W)}_{j,n}(s)\,ds\right] + O_p\left(n^{-1/2}\right) = -\mathsf{S}'_n(1) + O_p\left(n^{-1/2}\right).
\end{aligned}$$

Summing up, we obtain $\mathsf{S}_n(1) = O_p(n^{-1/2})$. Turning to $\mathsf{S}_n(2)$, we have

$$\begin{aligned}
&\mathsf{S}_n(2)\\
&= n\sum_{\tau_{j,n}\le t}\mathbb{E}_{\tau_{j-1,n}}\left[\int_{\tau_{j-1,n}}^{\tau_{j,n}}\left(\int_{\tau_{j-1,n}}^{s}\mu_u\,du\right)\sigma^2_s\,ds\right]\\
&= n\sum_{\tau_{j,n}\le t}\mu_{\tau_{j-1,n}}\,\sigma^2_{\tau_{j-1,n}}\,\mathbb{E}_{\tau_{j-1,n}}\left[\int_{\tau_{j-1,n}}^{\tau_{j,n}}\left(\int_{\tau_{j-1,n}}^{s}du\right)ds\right] + O_p\left(n^{-1/2}\right)\\
&= n\sum_{\tau_{j,n}\le t}\mu_{\tau_{j-1,n}}\,\sigma^2_{\tau_{j-1,n}}\,\mathbb{E}_{\tau_{j-1,n}}\left[\int_{\tau_{j-1,n}}^{\tau_{j,n}}\left(s - \tau_{j-1,n}\right)ds\right] + O_p\left(n^{-1/2}\right)\\
&= n\sum_{\tau_{j,n}\le t}\mu_{\tau_{j-1,n}}\,\sigma^2_{\tau_{j-1,n}}\,\mathbb{E}_{\tau_{j-1,n}}\left[\int_{\tau_{j-1,n}}^{\tau_{j,n}}\left(s - \tau_{j-1,n}\right)ds\right] + O_p\left(n^{-1/2}\right)\\
&= \frac{n}{2}\sum_{\tau_{j,n}\le t}\mu_{\tau_{j-1,n}}\,\sigma^2_{\tau_{j-1,n}}\,\mathbb{E}_{\tau_{j-1,n}}\left[\Delta^2_{j,n}\right] + O_p\left(n^{-1/2}\right)\\
&= \frac{n}{2}\sum_{\tau_{j,n}\le t}\mu_{\tau_{j-1,n}}\,\sigma^2_{\tau_{j-1,n}}\,\mathbb{E}_{\tau_{j-1,n}}\left[\frac{a^2}{n^2\,\sigma^4_{\tau_{j-1,n}}} + O_p\left(n^{-5/2}\right)\right] + O_p\left(n^{-1/2}\right)\\
&= \frac{1}{2n}\sum_{\tau_{j,n}\le t}\mu_{\tau_{j-1,n}}\,\frac{a^2}{\sigma^2_{\tau_{j-1,n}}} + O_p\left(n^{-1/2}\right) \xrightarrow{p} \frac{a}{2}\int_0^t \mu_s\,ds,
\end{aligned} \tag{101}$$

[16] Integration by parts applied to $M_t \doteq \int_{\tau_{j-1,n}}^{t}(\sigma_s - \sigma_{\tau_{j-1,n}})\,dW_s$ and $A_t \doteq t$ gives the result

$$M_{\tau_{j,n}}\,A_{\tau_{j,n}} = \int_{\tau_{j-1,n}}^{\tau_{j,n}}\left(\int_{\tau_{j-1,n}}^{s}\left(\sigma_u - \sigma_{\tau_{j-1,n}}\right)dW_u\right)ds + \int_{\tau_{j-1,n}}^{\tau_{j,n}} s\left(\sigma_s - \sigma_{\tau_{j-1,n}}\right)dW_s.$$

where the convergence in Eq. (85) was used. Next,

$$
\begin{aligned}
&\mathsf{S}_n\,(3)\\
&= n \sum_{\tau_{j,n}\le t} \mathbb{E}_{\tau_{j-1,n}}\left[\int_{\tau_{j-1,n}}^{\tau_{j,n}} \mu_s \left(\int_{\tau_{j-1,n}}^{s} \sigma_u\, dW_u\right)^2 ds\right]\\
&= n \sum_{\tau_{j,n}\le t} \mathbb{E}_{\tau_{j-1,n}}\left[\int_{\tau_{j-1,n}}^{\tau_{j,n}} \mu_{\tau_{j-1,n}} \left(\int_{\tau_{j-1,n}}^{s} \sigma_{\tau_{j-1,n}}\, dW_u\right)^2 ds\right] + O_p\left(n^{-1/2}\right)\\
&= n \sum_{\tau_{j,n}\le t} \mu_{\tau_{j-1,n}}\, \sigma^2_{\tau_{j-1,n}}\, \mathbb{E}_{\tau_{j-1,n}}\left[\int_{\tau_{j-1,n}}^{\tau_{j,n}} \left(W_s - W_{\tau_{j-1,n}}\right)^2 ds\right] + O_p\left(n^{-1/2}\right)\\
&= n \sum_{\tau_{j,n}\le t} \mu_{\tau_{j-1,n}}\, \sigma^2_{\tau_{j-1,n}} \left(\mathbb{E}_{\tau_{j-1,n}}\left[\left(\Delta_j^n W\right)^2 \Delta_{j,n}\right] - \frac{1}{2}\,\mathbb{E}_{\tau_{j-1,n}}\left[\Delta^2_{j,n}\right]\right)\\
&+ O_p\left(n^{-1/2}\right)\\
&= n \sum_{\tau_{j,n}\le t} \mu_{\tau_{j-1,n}}\, \sigma^2_{\tau_{j-1,n}} \left(\frac{a}{n\,\sigma^2_{\tau_{j-1,n}}}\,\mathbb{E}_{\tau_{j-1,n}}\left[\left(\Delta_j^n W\right)^2\right] - \frac{a^2}{2\,n^2\,\sigma^4_{\tau_{j-1,n}}}\right)\\
&+ O_p\left(n^{-1/2}\right)\\
&= n \sum_{\tau_{j,n}\le t} \mu_{\tau_{j-1,n}} \left(\frac{a}{n}\,\mathbb{E}_{\tau_{j-1,n}}\left[\Delta_{j,n}\right] - \frac{a^2}{2\,n^2\,\sigma^2_{\tau_{j-1,n}}}\right) + O_p\left(n^{-1/2}\right)\\
&= n \sum_{\tau_{j,n}\le t} \mu_{\tau_{j-1,n}} \left(\frac{a^2}{n^2\,\sigma^2_{\tau_{j-1,n}}} - \frac{a^2}{2\,n^2\,\sigma^2_{\tau_{j-1,n}}}\right) + O_p\left(n^{-1/2}\right)\\
&= \frac{a^2}{2\,n} \sum_{\tau_{j,n}\le t} \frac{\mu_{\tau_{j-1,n}}}{\sigma^2_{\tau_{j-1,n}}} + O_p\left(n^{-1/2}\right) \xrightarrow{\;p\;} \frac{a}{2}\int_0^t \mu_s\, ds,
\end{aligned}
\tag{102}
$$

using integration by parts[17]. Summing up, we now have

$$
\sqrt{n} \sum_{\tau_{j,n}\le t} \sqrt{n}\,\mathbb{E}_{\tau_{j-1,n}}\left[\left(\Delta_j^n X\right)^3\right] = 3\,\left(\mathsf{S}_n\,(1) + \mathsf{S}_n\,(2) + \mathsf{S}_n\,(3)\right) + O_p\left(n^{-1/2}\right) \xrightarrow{\;p\;} 3\,a\int_0^t \mu_s\, ds.
$$

To compute the asymptotic variance, consider that $(\Delta_j^n X)^6 = (\int_{\tau_{j-1,n}}^{\tau_{j,n}} \sigma_s\, dW_s)^6 + O_p(n^{-7/2})$. Using the sixth Bartlett

[17] Define $U_s = (W_s - W_{\tau_{j-1,n}})^2$ and $Z_s = s - \tau_{j-1,n}$. Therefore, $[U,Z]_s \equiv 0$ and integration by parts gives

$$
\begin{aligned}
\left(W_{\tau_{j,n}} - W_{\tau_{j-1,n}}\right)^2 \left(\tau_{j,n} - \tau_{j-1,n}\right) &= U_{\tau_{j,n}}\, Z_{\tau_{j,n}} = \int_{\tau_{j-1,n}}^{\tau_{j,n}} U_s\, dZ_s + \int_{\tau_{j-1,n}}^{\tau_{j,n}} Z_s\, dU_s\\
&= \int_{\tau_{j-1,n}}^{\tau_{j,n}} \left(W_s - W_{\tau_{j-1,n}}\right)^2 ds + \int_{\tau_{j-1,n}}^{\tau_{j,n}} \left(s - \tau_{j-1,n}\right) ds + 2\int_{\tau_{j-1,n}}^{\tau_{j,n}} \left(s - \tau_{j-1,n}\right)\left(W_s - W_{\tau_{j-1,n}}\right) dW_s.
\end{aligned}
$$

identity, we obtain

$$\begin{aligned}
n^2 \sum_{\tau_{j,n}\le t} \mathbb{E}_{\tau_{j-1,n}}\left[\left(\Delta_j^n X\right)^6\right] &= n^2 \sum_{\tau_{j,n}\le t}\left(15\,\mathbb{E}_{\tau_{j-1,n}}\left[\frac{a}{n}\left(\int_{\tau_{j-1,n}}^{\tau_{j,n}} \sigma_s\, dW_s\right)^4\right] - 45\,\mathbb{E}_{\tau_{j-1,n}}\left[\frac{a^2}{n^2}\left(\int_{\tau_{j-1,n}}^{\tau_{j,n}} \sigma_s\, dW_s\right)^2\right]\right) + \\
&\quad +15\,n^2 \sum_{\tau_{j,n}\le t} \frac{a^3}{n^3} + O_p\left(n^{-1/2}\right) \\
&= n^2 \sum_{\tau_{j,n}\le t}\left(15\,\frac{a}{n}\left(6\,\mathbb{E}_{\tau_{j-1,n}}\left[\frac{a}{n}\left(\int_{\tau_{j-1,n}}^{\tau_{j,n}} \sigma_s\, dW_s\right)^2\right] - 3\,\frac{a^2}{n^2}\right) - 45\,\frac{a^3}{n^3}\right) + \\
&\quad +15\,\frac{a^3}{n} \sum_{\tau_{j,n}\le t} 1 + O_p\left(n^{-1/2}\right) \\
&= n^2 \sum_{\tau_{j,n}\le t}\left(45\,\frac{a^3}{n^3} - 45\,\frac{a^3}{n^3}\right) + 15\,\frac{a^3}{n} \sum_{\tau_{j,n}\le t} 1 + O_p\left(n^{-1/2}\right) \xrightarrow{p} 15\,a^2 \int_0^t \sigma_s^2\, ds.
\end{aligned}$$

Finally, using the fourth Bartlett identity,

$$\begin{aligned}
n \sum_{\tau_{j,n}\le t} \mathbb{E}_{\tau_{j-1,n}}\left[\left(\Delta_j^n X\right)^3 \Delta_j^n W\right] &= n \sum_{\tau_{j,n}\le t} \mathbb{E}_{\tau_{j-1,n}}\left[\left(\int_{\tau_{j-1,n}}^{\tau_{j,n}} \sigma_s\, dW_s\right)^3 \Delta_j^n W\right] + O_p\left(n^{-1/2}\right) \\
&= n \sum_{\tau_{j,n}\le t} \sigma^3_{\tau_{j-1,n}}\, \mathbb{E}_{\tau_{j-1,n}}\left[\left(\Delta_j^n W\right)^4\right] + O_p\left(n^{-1/2}\right) \\
&= \frac{1}{n} \sum_{\tau_{j,n}\le t} \frac{3\,a^2}{\sigma_{\tau_{j-1,n}}} + O_p\left(n^{-1/2}\right) \xrightarrow{p} 3\,a \int_0^t \sigma_s\, ds.
\end{aligned}$$

□

Proof of Lemma 4. For $j \ge 1$, let $\Delta_{j,n} \doteq \tau_{j,n} - \tau_{j-1,n}$ and define, for $t \ge \tau_{j-1,n}$, the increment process $\delta_j^n\sigma(t) \doteq \sigma_t - \sigma_{\tau_{j-1,n}}$. Then, $\delta_j^n\sigma(\tau_{j-1,n}) = 0$ and, by continuity of σ, $\delta_j^n\sigma(\tau_{j,n}) \in \{U_n, -\ell_n\}$, $|\delta_j^n\sigma(t)| \le \max\{U_n, \ell_n\}$ for all $t \in [\tau_{j-1,n}, \tau_{j,n}]$. Define $\Gamma_t \doteq \sigma_t'^2 + \nu_t^2$. We have, on $[\tau_{j-1,n}, \tau_{j,n}]$,

$$d\left(\delta_j^n\sigma(t)\right) = \mu_t'\, dt + \sigma_t'\, dW_t + \nu_t\, dV_t, \quad d\left[\delta_j^n\sigma\right]_t = \Gamma_t\, dt.$$

Let $q_n(y) \doteq (U_n - y)(\ell_n + y) = U_n\ell_n + (U_n - \ell_n)y - y^2$ and define

$$u^{(n)}(y, s, v) \doteq \frac{q_n(y)}{s^2 + v^2}, \quad y, s, v \in \mathbb{R}, \quad s^2 + v^2 > 0.$$

We have the following boundary conditions

$$u^{(n)}\left(\delta_j^n\sigma\left(\tau_{j,n}\right), \sigma'_{\tau_{j,n}}, \nu_{\tau_{j,n}}\right) = 0 \quad \text{and} \quad u^{(n)}\left(\delta_j^n\sigma\left(\tau_{j-1,n}\right), \sigma'_{\tau_{j-1,n}}, \nu_{\tau_{j-1,n}}\right) = \frac{U_n\ell_n}{\Gamma_{\tau_{j-1,n}}} = \frac{ab}{n\,\Gamma_{\tau_{j-1,n}}}.$$

Also, writing $(y, s, v) \doteq (\delta_j^n\sigma(t), \sigma_t', \nu_t)$, $q_t \doteq q_n(\delta_j^n\sigma(t))$ and $\Gamma_t = s^2 + v^2$, we have the following partial derivatives

$$\begin{aligned}
u_y &= \frac{U_n - \ell_n - 2\,\delta_j^n\sigma(t)}{\Gamma_t}, \quad u_{yy} = -\frac{2}{\Gamma_t}, \quad u_s = u_{\sigma'} = -\frac{2\,\sigma_t'\,q_t}{\Gamma_t^2}, \quad u_v = u_\nu = -\frac{2\,\nu_t\,q_t}{\Gamma_t^2}, \\
u_{ss} &= -\frac{2\,q_t}{\Gamma_t^2} + \frac{8\,\sigma_t'^2\,q_t}{\Gamma_t^3}, \qquad u_{vv} = -\frac{2\,q_t}{\Gamma_t^2} + \frac{8\,\nu_t^2\,q_t}{\Gamma_t^3}, \quad u_{sv} = \frac{8\,\sigma_t'\nu_t\,q_t}{\Gamma_t^3}, \\
u_{ys} &= -\frac{2\,\sigma_t'\left(U_n - \ell_n - 2\delta_j^n\sigma(t)\right)}{\Gamma_t^2}, \qquad u_{yv} = -\frac{2\,\nu_t\left(U_n - \ell_n - 2\delta_j^n\sigma(t)\right)}{\Gamma_t^2}.
\end{aligned}$$

The quadratic covariations of interest are

$$\begin{aligned} d[\sigma']_t &= \left((\sigma_t'')^2 + (\nu_t'')^2 + \gamma_t^2\right) dt, \\ d[\nu]_t &= \left((\overline{\sigma}_t)^2 + (\overline{\nu}_t)^2 + (\overline{\gamma}_t)^2\right) dt, \\ d[\sigma', \nu]_t &= \left(\sigma_t''\overline{\sigma}_t + \nu_t''\overline{\nu}_t\right) dt, \\ d[\delta_j^n\sigma, \sigma']_t &= \left(\sigma_t'\sigma_t'' + \nu_t\nu_t''\right) dt, \\ d[\delta_j^n\sigma, \nu]_t &= \left(\sigma_t'\overline{\sigma}_t + \nu_t\overline{\nu}_t\right) dt. \end{aligned}$$

Let $\mathcal{Z}_t \doteq u^{(n)}(\delta_j^n\sigma(t), \sigma_t', \nu_t)$. Applying Itô formula,

$$\begin{aligned} d\mathcal{Z}_t = u_y\, d\left(\delta_j^n\sigma(t)\right) + u_s\, d\sigma_t' + u_v\, d\nu_t + \frac{1}{2}u_{yy}\, d[\delta_j^n\sigma]_t + \frac{1}{2}u_{ss}\, d[\sigma']_t + \frac{1}{2}u_{vv}\, d[\nu]_t \\ + u_{sv}\, d[\sigma', \nu]_t + u_{ys}\, d[\delta_j^n\sigma, \sigma']_t + u_{yv}\, d[\delta_j^n\sigma, \nu]_t. \end{aligned}$$

Since $(1/2)\,u_{yy}\, d[\delta_j^n\sigma]_t = (1/2)(-2/\Gamma_t)\Gamma_t\, dt = -dt$, we obtain, integrating from $\tau_{j-1,n}$ to $\tau_{j,n}$ (and using the boundary conditions $\mathcal{Z}_{\tau_{j,n}} = 0$ and $\mathcal{Z}_{\tau_{j-1,n}} = ab/(n\Gamma_{\tau_{j-1,n}})$), the following representation

$$\Delta_{j,n} = \frac{ab}{n\,\Gamma_{\tau_{j-1,n}}} + \int_{\tau_{j-1,n}}^{\tau_{j,n}} \Phi_{j,n}^{(W)}(t)\, dW_t + \int_{\tau_{j-1,n}}^{\tau_{j,n}} \Phi_{j,n}^{(V)}(t)\, dV_t + \int_{\tau_{j-1,n}}^{\tau_{j,n}} \Phi_{j,n}^{(Z)}(t)\, dZ_t + \int_{\tau_{j-1,n}}^{\tau_{j,n}} \Phi_{j,n}^{(B)}(t)\, dB_t + \int_{\tau_{j-1,n}}^{\tau_{j,n}} \Phi_{j,n}^{(\Delta)}(t)\, dt. \tag{103}$$

The stochastic-integral coefficients are

$$\begin{aligned} \Phi_{j,n}^{(W)}(t) &\doteq u_y\,\sigma_t' + u_s\,\sigma_t'' + u_v\,\overline{\sigma}_t, \quad \Phi_{j,n}^{(V)}(t) \doteq u_y\,\nu_t + u_s\,\nu_t'' + u_v\,\overline{\nu}_t, \\ \Phi_{j,n}^{(Z)}(t) &\doteq u_s\,\gamma_t, \quad \Phi_{j,n}^{(B)}(t) \doteq u_v\,\overline{\gamma}_t, \end{aligned}$$

and the drift coefficient is

$$\begin{aligned} \Phi_{j,n}^{(\Delta)}(t) \doteq\ & u_y\,\mu_t' + u_s\,\mu_t'' + u_v\,\overline{\mu}_t + \frac{1}{2}u_{ss}\left((\sigma_t'')^2 + (\nu_t'')^2 + \gamma_t^2\right) + \frac{1}{2}u_{vv}\left((\overline{\sigma}_t)^2 + (\overline{\nu}_t)^2 + (\overline{\gamma}_t)^2\right) \\ & + u_{sv}\left(\sigma_t''\overline{\sigma}_t + \nu_t''\overline{\nu}_t\right) + u_{ys}\left(\sigma_t'\sigma_t'' + \nu_t\nu_t''\right) + u_{yv}\left(\sigma_t'\overline{\sigma}_t + \nu_t\overline{\nu}_t\right). \end{aligned}$$

On $[\tau_{j-1,n}, \tau_{j,n}]$, we have $\delta_j^n\sigma(t) = O_p(n^{-1/2})$ uniformly and, hence,

$$q_t = q_n\left(\delta_j^n\sigma(t)\right) = O_p\left(n^{-1}\right), \quad U_n - \ell_n - 2\delta_j^n\sigma(t) = O_p\left(n^{-1/2}\right),$$

uniformly. Under boundedness of the coefficients and Γ_t bounded away from 0, we have

$$u_y = O_p(n^{-1/2}), \qquad u_s = O_p(n^{-1}), \qquad u_v = O_p(n^{-1}), \tag{104}$$

$$u_{ss} = O_p(n^{-1}), \quad u_{vv} = O_p(n^{-1}), \quad u_{sv} = O_p(n^{-1}), \quad u_{ys} = O_p(n^{-1/2}), \quad u_{yv} = O_p(n^{-1/2}),$$

uniformly on $[\tau_{j-1,n}, \tau_{j,n}]$. Consequently,

$$\Phi_{j,n}^{(W)}(t) = O_p(n^{-1/2}), \quad \Phi_{j,n}^{(V)}(t) = O_p(n^{-1/2}), \quad \Phi_{j,n}^{(Z)}(t) = O_p(n^{-1}), \quad \Phi_{j,n}^{(B)}(t) = O_p(n^{-1}), \quad \Phi_{j,n}^{(\Delta)}(t) = O_p(n^{-1/2}),$$

uniformly, and the leading term implies $\Delta_{j,n} = O_p(n^{-1})$. Hence, per interval,

$$\int_{\tau_{j-1,n}}^{\tau_{j,n}} \Phi_{j,n}^{(W)}(t)\, dW_t = O_p(n^{-1}), \quad \int_{\tau_{j-1,n}}^{\tau_{j,n}} \Phi_{j,n}^{(V)}(t)\, dV_t = O_p(n^{-1}),$$

$$\int_{\tau_{j-1,n}}^{\tau_{j,n}} \Phi_{j,n}^{(Z)}(t)\, dZ_t = O_p(n^{-3/2}), \quad \int_{\tau_{j-1,n}}^{\tau_{j,n}} \Phi_{j,n}^{(B)}(t)\, dB_t = O_p(n^{-3/2}), \quad \int_{\tau_{j-1,n}}^{\tau_{j,n}} \Phi_{j,n}^{(\Delta)}(t)\, dt = O_p(n^{-3/2}).$$

□

Before proceeding with the proof of Theorem 3.4, we state and prove some useful lemmas.

Lemma 11. *Let B be a standard Brownian motion ($B_0 = 0$). Given constants $a > 0$ and $b > 0$, consider*

$$\tau = \inf\{t \geq 0 \mid B_t \notin (-b, a)\}.$$

We have the following identities

$$\mathbb{E}\left[\int_0^\tau \left(B_s^2 - s\right)\, ds\right] \;=\; -\frac{2}{3}a^2\, b^2, \tag{105}$$

$$\mathbb{E}\left[\int_0^\tau B_s\, ds\right] \;=\; \frac{a\, b\, (a-b)}{3}, \tag{106}$$

Proof. Set $N_t \doteq B_t^2 - t$. Then N is a martingale. By integration by parts,

$$d\,(t\, N_t) = N_t\, dt + t\, dN_t \quad\Longrightarrow\quad t\, N_t = \int_0^t N_s\, ds + \int_0^t s\, dN_s,$$

having used $N_0 = 0$. Rearranging,

$$\int_0^t \left(B_s^2 - s\right)\, ds - t\, \left(B_t^2 - t\right) = -\int_0^t s\, dN_s,$$

which is a stochastic integral and, hence, a (local) martingale. Therefore,

$$M_t \doteq \int_0^t \left(B_s^2 - s\right)\, ds - t\, \left(B_t^2 - t\right)$$

is a (local) martingale. Applying optional stopping gives

$$\mathbb{E}\left[M_\tau\right] = 0 \quad\Longrightarrow\quad \mathbb{E}\left[\int_0^\tau (B_s^2 - s)\, ds\right] = \mathbb{E}\left[\tau\, (B_\tau^2 - \tau)\right].$$

Thus, it remains to compute $\mathbb{E}[\tau B_\tau^2] - \mathbb{E}[\tau^2]$. Since $B_\tau \in \{a, -b\}$, optional stopping for B_t gives

$$0 = \mathbb{E}\left[B_\tau\right] = a\, \mathbb{P}\left[B_\tau = a\right] - b\, \mathbb{P}\left[B_\tau = -b\right] \quad\Longrightarrow\quad \mathbb{P}\left[B_\tau = a\right] = \frac{b}{a+b}, \qquad \mathbb{P}\left[B_\tau = -b\right] = \frac{a}{a+b}.$$

Optional stopping for $B_t^2 - t$, instead, yields

$$0 = \mathbb{E}\left[B_\tau^2 - \tau\right] \quad\Longrightarrow\quad \mathbb{E}\left[\tau\right] = \mathbb{E}\left[B_\tau^2\right] = \frac{b}{a+b}\, a^2 + \frac{a}{a+b}\, b^2 = a\, b.$$

Stopping the Brownian polynomial martingale $B_t^3 - 3\, t\, B_t$ (i.e., essentially the third Bartlett identity) gives

$$0 = \mathbb{E}\left[B_\tau^3 - 3\tau\, B_\tau\right] \quad\Longrightarrow\quad \mathbb{E}\left[\tau\, B_\tau\right] = \frac{1}{3}\mathbb{E}\left[B_\tau^3\right] = \frac{1}{3}\left(\frac{b}{a+b}\, a^3 + \frac{a}{a+b}\, (-b)^3\right) = \frac{b\, a^3 - a\, b^3}{3(a+b)} = \frac{a\, b\, (a-b)}{3}.$$

Since $B_\tau \in \{a, -b\}$, B_τ^2 is an affine function[18] of B_τ. In fact, $B_\tau^2 = ab + (a-b)B_\tau$. Hence,

$$\mathbb{E}\left[\tau B_\tau^2\right] = a\, b\, \mathbb{E}\left[\tau\right] + (a-b)\, \mathbb{E}\left[\tau B_\tau\right] = a^2\, b^2 + (a-b)\, \frac{a\, b\, (a-b)}{3} = \frac{a\, b\, (a^2 + a\, b + b^2)}{3}. \tag{107}$$

Stopping the Brownian polynomial martingale $B_t^4 - 6\, t\, B_t^2 + 3\, t^2$ (i.e., essentially the fourth Bartlett identity) gives

$$0 = \mathbb{E}\left[B_\tau^4 - 6\, \tau B_\tau^2 + 3\, \tau^2\right] \quad\Longrightarrow\quad 3\mathbb{E}\left[\tau^2\right] = 6\mathbb{E}\left[\tau\, B_\tau^2\right] - \mathbb{E}\left[B_\tau^4\right].$$

Also,

$$\mathbb{E}\left[B_\tau^4\right] = \frac{b}{a+b}\, a^4 + \frac{a}{a+b}\, b^4 = \frac{b\, a^4 + a\, b^4}{a+b} = a\, b\, \left(a^2 - a\, b + b^2\right),$$

which, together with Eq. (107), yields

$$\mathbb{E}\left[\tau^2\right] = 2\, \frac{a\, b\, (a^2 + ab + b^2)}{3} - \frac{a\, b\, (a^2 - a\, b + b^2)}{3} = \frac{a\, b\, (a^2 + 3\, a\, b + b^2)}{3}.$$

[18] The identity $B_\tau^2 = ab + (a-b)B_\tau$ is verified directly at both possible values $B_\tau = a$ and $B_\tau = -b$.

Summing up, we obtain

$$\mathbb{E}\left[\int_0^\tau (B_s^2 - s)\, ds\right] = \mathbb{E}\left[\tau\, B_\tau^2\right] - \mathbb{E}\left[\tau^2\right] = \frac{a\,b\,\left(a^2 + a\,b + b^2\right)}{3} - \frac{a\,b\,\left(a^2 + 3\,a\,b + b^2\right)}{3} = -\frac{2}{3}\,a^2\,b^2,$$

which is Eq. (105). To prove Eq. (106), we follow a similar logic. First, from $d(tB_t) = B_t\,dt + t\,dB_t$ we derive $\int_0^t B_s\,ds - tB_t = -\int_0^t s\,dB_s$, which implies that $M_t \doteq \int_0^t B_s\,ds - tB_t$ is a martingale. Optional stopping gives

$$\mathbb{E}\left[\int_0^\tau B_s\,ds\right] = \mathbb{E}\left[\tau B_\tau\right] = \frac{1}{3}\,\mathbb{E}\left[B_\tau^3\right] = \frac{1}{3}\left(a^3\frac{b}{a+b} + (-b)^3\frac{a}{a+b}\right) = \frac{ab(a^2-b^2)}{3(a+b)} = \frac{ab(a-b)}{3},$$

where, in the second equation, we have used the polynomial martingale $B_t^3 - 3tB_t$ (i.e., essentially the third Bartlett identity). □

Lemma 12. *Let Assumptions 1, 2, 6, and 7 hold. For all n and every $j \geq 1$, we have the following expansions*

$$\mathbb{E}_{\tau_{j-1,n}}\left[\int_{\tau_{j-1,n}}^{\tau_{j,n}}\left(\int_{\tau_{j-1,n}}^{t}\left(V_s - V_{\tau_{j-1,n}}\right)dW_s\right)dt\right] = -\frac{a^2b^2}{3}\,\frac{\sigma'_{\tau_{j-1,n}}\nu_{\tau_{j-1,n}}}{\Gamma^3_{\tau_{j-1,n}}}\,\frac{1}{n^2} + o_p\left(n^{-2}\right), \tag{108}$$

$$\mathbb{E}_{\tau_{j-1,n}}\left[\int_{\tau_{j-1,n}}^{\tau_{j,n}}\left(\int_{\tau_{j-1,n}}^{t}\left(W_s - W_{\tau_{j-1,n}}\right)dW_s\right)dt\right] = -\frac{a^2b^2}{3}\,\frac{\sigma'^2_{\tau_{j-1,n}}}{\Gamma^3_{\tau_{j-1,n}}}\,\frac{1}{n^2} + o_p(n^{-2}), \tag{109}$$

$$\mathbb{E}_{\tau_{j-1,n}}\left[\int_{\tau_{j-1,n}}^{\tau_{j,n}}\left(\int_{\tau_{j-1,n}}^{t}\left(Z_s - Z_{\tau_{j-1,n}}\right)dW_s\right)dt\right] = o_p(n^{-2}). \tag{110}$$

Also, under the (simplifying) assumptions in Eq. (22), we have

$$\mathbb{E}_{\tau_{j-1,n}}\left[\int_{\tau_{j-1,n}}^{\tau_{j,n}}\Phi^{(\Delta)}_{j,n}(s)\,ds\right] = \mu'_{\tau_{j-1,n}}\,\frac{a\,b\,(a-b)}{3\,n^{3/2}\,\Gamma^2_{\tau_{j-1,n}}} + O_p\left(n^{-2}\right), \tag{111}$$

$$\mathbb{E}_{\tau_{j-1,n}}\left[\int_{\tau_{j-1,n}}^{\tau_{j,n}}\left(W_t - W_{\tau_{j-1,n}}\right)dt\right] = \sigma'_{\tau_{j-1,n}}\left(\frac{ab(a-b)}{3\,\Gamma^2_{\tau_{j-1,n}}\,n^{3/2}} - \frac{2a^2b^2}{3}\,\frac{\mu'_{\tau_{j-1,n}}}{\Gamma^3_{\tau_{j-1,n}}\,n^2}\right) + O_p(n^{-5/2}), \tag{112}$$

$$\mathbb{E}_{\tau_{j-1,n}}\left[\left(\Delta_j^n W\right)^2\int_{\tau_{j-1,n}}^{\tau_{j,n}}(\sigma_s - \sigma_{\tau_{j-1,n}})\,dW_s\right] = \frac{ab\,(a-b)^2}{3}\,\frac{\sigma'_{\tau_{j-1,n}}}{n^2\,\Gamma^2_{\tau_{j-1,n}}} + O_P(n^{-5/2}), \tag{113}$$

$$\mathbb{E}\left[\left(\Delta_j^n W\right)^2\Delta_{j,n}\,\middle|\,\mathcal{F}_{\tau_{j-1,n}}\right] = \frac{ab(a^2+3ab+b^2)}{3\,n^2\Gamma^2_{\tau_{j-1,n}}} - \frac{2a^2b^2}{3}\,\frac{\sigma'^2_{\tau_{j-1,n}}}{n^2\Gamma^3_{\tau_{j-1,n}}} + O_p(n^{-5/2}). \tag{114}$$

Proof. Fix $n \in \mathbb{N}$ and $j \geq 1$. For $t \geq \tau_{j-1,n}$ define

$$U_{j,n}(t) \doteq \frac{\sigma'_{\tau_{j-1,n}}(W_t - W_{\tau_{j-1,n}}) + \nu_{\tau_{j-1,n}}(V_t - V_{\tau_{j-1,n}})}{\Gamma^{1/2}_{\tau_{j-1,n}}}, \qquad R_{j,n}(t) \doteq \frac{\nu_{\tau_{j-1,n}}(W_t - W_{\tau_{j-1,n}}) - \sigma'_{\tau_{j-1,n}}(V_t - V_{\tau_{j-1,n}})}{\sqrt{\Gamma_{\tau_{j-1,n}}}}. \tag{115}$$

Since $(W_{\tau_{j-1,n}+s} - W_{\tau_{j-1,n}}, V_{\tau_{j-1,n}+s} - V_{\tau_{j-1,n}})_{s\geq 0}$ is a 2D Brownian motion independent of $\mathcal{F}_{\tau_{j-1,n}}$ and the transformation in Eq. (115) is orthogonal, $(U_{j,n}(\tau_{j-1,n}+s), R_{j,n}(\tau_{j-1,n}+s))_{s\geq 0}$ is a 2D Brownian motion independent of $\mathcal{F}_{\tau_{j-1,n}}$. In particular, these two elapsed-time processes are independent standard Brownian motions. Let

$$\epsilon_{j,n}(t) \doteq \sigma_t - \sigma_{\tau_{j-1,n}} - \Gamma^{1/2}_{\tau_{j-1,n}}\,U_{j,n}(t), \qquad t \geq \tau_{j-1,n}.$$

From Assumption 1,

$$\epsilon_{j,n}(t) = \int_{\tau_{j-1,n}}^t \mu'_s\,ds + \int_{\tau_{j-1,n}}^t (\sigma'_s - \sigma'_{\tau_{j-1,n}})\,dW_s + \int_{\tau_{j-1,n}}^t (\nu_s - \nu_{\tau_{j-1,n}})\,dV_s.$$

Using that $\Delta_{j,n} \doteq \tau_{j,n} - \tau_{j-1,n} = O_p(n^{-1})$ (e.g. by Lemma 4), the boundedness of μ', σ', ν (Assumption 1), the Itô

semimartingale structure of σ' and ν (Assumption 7), and BDG inequality yield

$$\sup_{t\in[\tau_{j-1,n},\tau_{j,n}]}\left|\epsilon_{j,n}(t)\right| = O_p(n^{-1}). \tag{116}$$

For $\varepsilon > 0$, define the two perturbed exit times (recall that $U_{j,n}(\tau_{j-1,n}) = 0$)

$$\begin{aligned}
\Delta^{(+)}_{j,n}(\varepsilon) &\doteq \inf\left\{s>0:\ \Gamma^{1/2}_{\tau_{j-1,n}}U_{j,n}\left(\tau_{j-1,n}+s\right)\notin\left(-\frac{b}{\sqrt{n}}-\frac{\varepsilon}{n},\frac{a}{\sqrt{n}}+\frac{\varepsilon}{n}\right)\right\},\\
\Delta^{(-)}_{j,n}(\varepsilon) &\doteq \inf\left\{s>0:\ \Gamma^{1/2}_{\tau_{j-1,n}}U_{j,n}\left(\tau_{j-1,n}+s\right)\notin\left(-\frac{b}{\sqrt{n}}+\frac{\varepsilon}{n},\frac{a}{\sqrt{n}}-\frac{\varepsilon}{n}\right)\right\},
\end{aligned}$$

and the unperturbed exit time for the Brownian motion $U_{j,n}$:

$$\Delta^{(0)}_{j,n} \doteq \inf\left\{s>0:\ \Gamma^{1/2}_{\tau_{j-1,n}}U_{j,n}\left(\tau_{j-1,n}+s\right)\notin\left(-b/\sqrt{n},\ a/\sqrt{n}\right)\right\}.$$

By inclusion of intervals,

$$\Delta^{(-)}_{j,n}(\varepsilon) \le \Delta^{(0)}_{j,n} \le \Delta^{(+)}_{j,n}(\varepsilon) \qquad \text{for every } \varepsilon>0. \tag{117}$$

Now, introduce the event

$$A_{j,n}(\varepsilon) \doteq \left\{\sup_{t\in[\tau_{j-1,n},\tau_{j,n}]}\left|\epsilon_{j,n}(t)\right| \le \frac{\varepsilon}{n}\right\}.$$

On $A_{j,n}(\varepsilon)$, the stopping rule defining $\tau_{j,n}$ and the bound $|\epsilon_{j,n}| \le \varepsilon/n$ imply the sandwich

$$\Delta^{(-)}_{j,n}(\varepsilon) \le \Delta_{j,n} \le \Delta^{(+)}_{j,n}(\varepsilon). \tag{118}$$

Combining Eq. (117) and Eq. (118) yields, on $A_{j,n}(\varepsilon)$,

$$\left|\Delta_{j,n}-\Delta^{(0)}_{j,n}\right| \le \Delta^{(+)}_{j,n}(\varepsilon)-\Delta^{(-)}_{j,n}(\varepsilon). \tag{119}$$

In order to prove that

$$\Delta^{(+)}_{j,n}(\varepsilon)-\Delta^{(-)}_{j,n}(\varepsilon) = O_p\left(\frac{\varepsilon}{n^{3/2}}\right), \tag{120}$$

define the rescaled Brownian motion

$$B_s \doteq \left(n\Gamma_{\tau_{j-1,n}}\right)^{1/2}U_{j,n}\left(\tau_{j-1,n}+\frac{s}{n\Gamma_{\tau_{j-1,n}}}\right) \qquad s\ge 0. \tag{121}$$

Then, $\{B_s\}_{s\ge0}$ is a standard Brownian motion independent of $\mathcal{F}_{\tau_{j-1,n}}$. Let $\delta_n \doteq \varepsilon/\sqrt{n}$ and set the rescaled exit times

$$\begin{aligned}
\tau^{(+)}_B(\delta_n) &\doteq \inf\{s>0:\ B_s\notin(-b-\delta_n,\ a+\delta_n)\},\\
\tau^{(-)}_B(\delta_n) &\doteq \inf\{s>0:\ B_s\notin(-b+\delta_n,\ a-\delta_n)\}.
\end{aligned} \tag{122}$$

A direct change of variables gives the identities

$$\Delta^{(+)}_{j,n}(\varepsilon) = \frac{\tau^{(+)}_B(\delta_n)}{n\Gamma_{\tau_{j-1,n}}}, \quad \Delta^{(-)}_{j,n}(\varepsilon) = \frac{\tau^{(-)}_B(\delta_n)}{n\Gamma_{\tau_{j-1,n}}}.$$

Hence, it suffices to show $\tau^{(+)}_B(\delta_n)-\tau^{(-)}_B(\delta_n) = O_p(\delta_n)$. To simplify notation, let $\tau^- \doteq \tau^{(-)}_B(\delta_n)$ and $\tau^+ \doteq \tau^{(+)}_B(\delta_n)$. Since $(-b+\delta_n,\ a-\delta_n)\subset(-b-\delta_n,\ a+\delta_n)$, we have $\tau^-\le\tau^+$ and, by continuity,

$$B_{\tau^-}\in\{-b+\delta_n, a-\delta_n\}.$$

By the strong Markov property[19] $\tau^+-\tau^-$ is stochastically dominated[20] by an exit time from $(-b-\delta_n,\ a+\delta_n)$ starting at either endpoint $a-\delta_n$ or $-b+\delta_n$. For Brownian motion, the expected exit time from (ℓ,u) starting at

[19] If B is a standard Brownian motion and τ is a stopping time, then $\{B_{t+\tau}-B_\tau\}_{t\ge0}$ is (i) independent of $\mathcal{F}_\tau$ and (ii) a standard Brownian motion.

[20] In the sense that, conditionally on $B_{\tau^-}=x$ for a given $x\in\{-b+\delta_n, a-\delta_n\}$, the interval $\tau^+-\tau^-$ is an exit time from the big interval

$x \in (\ell, u)$ equals $(x-\ell)(u-x)$. Therefore,

$$\begin{aligned}
\mathbb{E}\left[\tau^+ - \tau^-\right] &= \mathbb{E}\left[\tau^+ - \tau^- \,\middle|\, B_{\tau^-} = -b + \delta_n\right] \mathbb{P}\left[B_{\tau^-} = -b + \delta_n\right] + \mathbb{E}\left[\tau^+ - \tau^- \,\middle|\, B_{\tau^-} = a - \delta_n\right] \mathbb{P}\left[B_{\tau^-} = a - \delta_n\right] \\
&\leq \max\left\{\mathbb{E}\left[\tau^+ - \tau^- \,\middle|\, B_{\tau^-} = -b + \delta_n\right], \mathbb{E}\left[\tau^+ - \tau^- \,\middle|\, B_{\tau^-} = a - \delta_n\right]\right\} \\
&= \max\left\{(-b + \delta_n + b + \delta_n)(a + \delta_n - (-b + \delta_n)), (a - \delta_n + b + \delta_n)(a + \delta_n - (a - \delta_n))\right\} \\
&= 2(a+b)\delta_n.
\end{aligned}$$

By Markov's inequality, for any $M > 0$,

$$\mathbb{P}\left[\tau^+ - \tau^- > M\delta_n\right] \leq \frac{\mathbb{E}\left[\tau^+ - \tau^-\right]}{M\delta_n} \leq \frac{2(a+b)}{M},$$

which proves $\tau^+ - \tau^- = O_p(\delta_n) = O_p(\varepsilon\, n^{-1/2})$ and, thus, Eq. (120). Fix now $\eta > 0$ and $\varepsilon > 0$. By[21] Eq. (119) and Eq. (120),

$$\limsup_{n\to\infty} \mathbb{P}\left[\left|\Delta_{j,n} - \Delta_{j,n}^{(0)}\right| > \frac{\eta}{n}\right] \leq \limsup_{n\to\infty} \mathbb{P}\left[A_{j,n}(\varepsilon)^c\right] + \limsup_{n\to\infty} \mathbb{P}\left[\Delta_{j,n}^{(+)}(\varepsilon) - \Delta_{j,n}^{(-)}(\varepsilon) > \frac{\eta}{n}\right] = \limsup_{n\to\infty} \mathbb{P}\left[A_{j,n}(\varepsilon)^c\right],$$

because $\Delta_{j,n}^{(+)}(\varepsilon) - \Delta_{j,n}^{(-)}(\varepsilon) = O_p(\varepsilon n^{-3/2})$ implies that the second probability tends to 0.

Moreover, Eq. (116) implies that $n \sup_{t\in[\tau_{j-1,n},\tau_{j,n}]} |\epsilon_{j,n}(t)|$ is tight, hence

$$\lim_{\varepsilon\to\infty} \limsup_{n\to\infty} \mathbb{P}\left[A_{j,n}(\varepsilon)^c\right] = 0.$$

Letting $\varepsilon \to \infty$ yields that $\lim_{n\to\infty} \mathbb{P}[|\Delta_{j,n} - \Delta_{j,n}^{(0)}| > \frac{\eta}{n}]$ exists and that

$$\lim_{n\to\infty} \mathbb{P}\left[\left|\Delta_{j,n} - \Delta_{j,n}^{(0)}\right| > \frac{\eta}{n}\right] = 0,$$

that is,

$$\Delta_{j,n} - \Delta_{j,n}^{(0)} = o_p\left(n^{-1}\right). \tag{123}$$

We need to evaluate $\mathbb{E}_{\tau_{j-1,n}}[\int_{\tau_{j-1,n}}^{\tau_{j-1,n}+\Delta_{j,n}} M_t\, dt]$, with M_t a $\mathcal{F}_t$-martingale of the form $M_t = \int_{\tau_{j-1,n}}^t (V_s - V_{\tau_{j-1,n}})\, dW_s$ or $M_t = \int_{\tau_{j-1,n}}^t (W_s - W_{\tau_{j-1,n}})\, dW_s$, which means that, in both cases, $[M]_t = O_p(n^{-2})$ and BDG inequality gives

$$\sup_{t\in\left[\tau_{j-1,n},\tau_{j-1,n}+\left(\Delta_{j,n}\vee\Delta_{j,n}^{(0)}\right)\right]} |M_t| = O_p\left(n^{-1}\right). \tag{124}$$

Define, for generic Δ, the integral

$$\mathcal{I}(\Delta) \doteq \int_{\tau_{j-1,n}}^{\tau_{j-1,n}+\Delta} M_t\, dt.$$

$(-b-\delta_n, a+\delta_n)$. Unconditionally, its law is a mixture of those two cases, that is

$$\mathbb{E}\left[\tau^+ - \tau^-\right] = \mathbb{P}\left[B_{\tau^-} = -b + \delta_n\right] \mathbb{E}\left[\tau^+ - \tau^- \,\middle|\, B_{\tau^-} = -b + \delta_n\right] + \mathbb{P}\left[B_{\tau^-} = a - \delta_n\right] \mathbb{E}\left[\tau^+ - \tau^- \,\middle|\, B_{\tau^-} = a - \delta_n\right].$$

[21] To better understand the steps that follow, consider the events $E \doteq \{|\Delta_{j,n} - \Delta_{j,n}^{(0)}| > \eta/n\}$ and $A \doteq A_{j,n}(\varepsilon)$. Then, $E = (E \cap A^c) \cup (E \cap A)$. Hence, by the union bound,

$$\mathbb{P}[E] \leq \mathbb{P}[E \cap A^c] + \mathbb{P}[E \cap A] \leq \mathbb{P}[A^c] + \mathbb{P}[E \cap A].$$

On the event A, Eq. (119) yields

$$E \cap A \subseteq \left\{\Delta_{j,n}^{(+)}(\varepsilon) - \Delta_{j,n}^{(-)}(\varepsilon) > \frac{\eta}{n}\right\},$$

and, consequently,

$$\mathbb{P}[E \cap A] \leq \mathbb{P}\left[\Delta_{j,n}^{(+)}(\varepsilon) - \Delta_{j,n}^{(-)}(\varepsilon) > \frac{\eta}{n}\right].$$

Combining the previous bounds gives

$$\mathbb{P}\left[\left|\Delta_{j,n} - \Delta_{j,n}^{(0)}\right| > \frac{\eta}{n}\right] \leq \mathbb{P}\left[A_{j,n}(\varepsilon)^c\right] + \mathbb{P}\left[\Delta_{j,n}^{(+)}(\varepsilon) - \Delta_{j,n}^{(-)}(\varepsilon) > \frac{\eta}{n}\right].$$

Then, pathwise, we have

$$\begin{aligned}\left|\mathcal{I}\left(\Delta_{j,n}\right)-\mathcal{I}\left(\Delta_{j,n}^{(0)}\right)\right| &= \left|\int_{\tau_{j-1,n}+\Delta_{j,n}^{(0)}}^{\tau_{j-1,n}+\Delta_{j,n}} M_t\,dt\right| \leq \int_{\tau_{j-1,n}+\left(\Delta_{j,n}^{(0)}\wedge\Delta_{j,n}\right)}^{\tau_{j-1,n}+\left(\Delta_{j,n}^{(0)}\vee\Delta_{j,n}\right)} |M_t|\,dt \\ &\leq \left|\Delta_{j,n}-\Delta_{j,n}^{(0)}\right| \sup_{t\in\left[\tau_{j-1,n}+\left(\Delta_{j,n}\wedge\Delta_{j,n}^{(0)}\right),\tau_{j-1,n}+\left(\Delta_{j,n}\vee\Delta_{j,n}^{(0)}\right)\right]}|M_t| \\ &\leq \left|\Delta_{j,n}-\Delta_{j,n}^{(0)}\right| \sup_{t\in\left[\tau_{j-1,n},\tau_{j-1,n}+\left(\Delta_{j,n}\vee\Delta_{j,n}^{(0)}\right)\right]}|M_t| \end{aligned} \tag{125}$$

Using Eq. (124), Eq. (123) now gives

$$\int_{\tau_{j-1,n}}^{\tau_{j-1,n}+\Delta_{j,n}} M_t\,dt = \int_{\tau_{j-1,n}}^{\tau_{j-1,n}+\Delta_{j,n}^{(0)}} M_t\,dt + o_p(n^{-2}), \tag{126}$$

a property that we are going to use in what follows. Now, for the rescaled Brownian motion defined in Eq. (121), consider the stopping time

$$\tau_B \doteq \inf\{t>0:\ B_t\notin(-b,a)\}. \tag{127}$$

Brownian scaling gives that

$$\Delta_{j,n}^{(0)} = \frac{\tau_B}{n\Gamma_{\tau_{j-1,n}}}. \tag{128}$$

Inverting Eq. (115) yields, for $t\geq\tau_{j-1,n}$,

$$W_t - W_{\tau_{j-1,n}} = \frac{\sigma'_{\tau_{j-1,n}}U_{j,n}(t)+\nu_{\tau_{j-1,n}}R_{j,n}(t)}{\Gamma_{\tau_{j-1,n}}^{1/2}}, \qquad V_t - V_{\tau_{j-1,n}} = \frac{\nu_{\tau_{j-1,n}}U_{j,n}(t)-\sigma'_{\tau_{j-1,n}}R_{j,n}(t)}{\Gamma_{\tau_{j-1,n}}^{1/2}}. \tag{129}$$

Hence, for $t\in[\tau_{j-1,n},\tau_{j,n}]$,

$$\begin{aligned}\int_{\tau_{j-1,n}}^{t}\left(V_s-V_{\tau_{j-1,n}}\right)dW_s &= \frac{\sigma'_{\tau_{j-1,n}}\nu_{\tau_{j-1,n}}}{\Gamma_{\tau_{j-1,n}}}\int_{\tau_{j-1,n}}^{t}U_{j,n}(s)\,dU_{j,n}(s) - \frac{\sigma'^2_{\tau_{j-1,n}}}{\Gamma_{\tau_{j-1,n}}}\int_{\tau_{j-1,n}}^{t}R_{j,n}(s)\,dU_{j,n}(s) \\ &\quad + \frac{\nu^2_{\tau_{j-1,n}}}{\Gamma_{\tau_{j-1,n}}}\int_{\tau_{j-1,n}}^{t}U_{j,n}(s)\,dR_{j,n}(s) - \frac{\sigma'_{\tau_{j-1,n}}\nu_{\tau_{j-1,n}}}{\Gamma_{\tau_{j-1,n}}}\int_{\tau_{j-1,n}}^{t}R_{j,n}(s)\,dR_{j,n}(s).\end{aligned} \tag{130}$$

Let $\mathcal{G}_{j,n}$ be the auxiliary σ-algebra

$$\mathcal{G}_{j,n} \doteq \mathcal{F}_{\tau_{j-1,n}} \vee \sigma\left\{U_{j,n}\left(\tau_{j-1,n}+s\right)\,\middle|\, s\geq 0\right\}.$$

Then, $\Delta_{j,n}^{(0)}$ is $\mathcal{G}_{j,n}$-measurable and $\{R_{j,n}(\tau_{j-1,n}+s)\}_{s\geq0}$ is a Brownian motion independent of $\mathcal{G}_{j,n}$ such that $R_{j,n}(\tau_{j-1,n})=0$. Thus,

$$\begin{aligned}\mathbb{E}_{\tau_{j-1,n}}\left[\int_{\tau_{j-1,n}}^{\tau_{j,n}}\int_{\tau_{j-1,n}}^{t}R_{j,n}(s)\,dU_{j,n}(s)\,dt\right] &= \mathbb{E}_{\tau_{j-1,n}}\left[\int_{\tau_{j-1,n}}^{\tau_{j-1,n}+\Delta_{j,n}^{(0)}}\int_{\tau_{j-1,n}}^{t}R_{j,n}(s)\,dU_{j,n}(s)\,dt\right]+o_p\left(n^{-2}\right) \\ &= \mathbb{E}_{\tau_{j-1,n}}\left[\mathbb{E}\left[\int_{\tau_{j-1,n}}^{\tau_{j-1,n}+\Delta_{j,n}^{(0)}}\int_{\tau_{j-1,n}}^{t}R_{j,n}(s)\,dU_{j,n}(s)\,dt\,\middle|\,\mathcal{G}_{j,n}\right]\right]+o_p\left(n^{-2}\right) \\ &= \mathbb{E}_{\tau_{j-1,n}}\left[\int_{\tau_{j-1,n}}^{\tau_{j-1,n}+\Delta_{j,n}^{(0)}}\mathbb{E}\left[\int_{\tau_{j-1,n}}^{t}R_{j,n}(s)\,dU_{j,n}(s)\,\middle|\,\mathcal{G}_{j,n}\right]dt\right]+o_p\left(n^{-2}\right) \\ &= o_p\left(n^{-2}\right).\end{aligned}$$

The same reasoning applies to all the integrals in Eq. (130) except for the first one. Therefore,

$$\begin{aligned}
&\mathbb{E}_{\tau_{j-1,n}}\left[\int_{\tau_{j-1,n}}^{\tau_{j,n}}\left(\int_{\tau_{j-1,n}}^{t}\left(V_s - V_{\tau_{j-1,n}}\right)dW_s\right)dt\right] = \\
&= \frac{\sigma'_{\tau_{j-1,n}}\nu_{\tau_{j-1,n}}}{\Gamma_{\tau_{j-1,n}}}\mathbb{E}_{\tau_{j-1,n}}\left[\int_{\tau_{j-1,n}}^{\tau_{j,n}}\left(\int_{\tau_{j-1,n}}^{t}U_{j,n}(s)\,dU_{j,n}(s)\right)dt\right] + o_p\left(\frac{1}{n^2}\right) \\
&= \frac{\sigma'_{\tau_{j-1,n}}\nu_{\tau_{j-1,n}}}{2\,\Gamma_{\tau_{j-1,n}}}\mathbb{E}_{\tau_{j-1,n}}\left[\int_{\tau_{j-1,n}}^{\tau_{j,n}}\left(U_{j,n}(t)^2 - (t-\tau_{j-1,n})\right)dt\right] + o_p\left(\frac{1}{n^2}\right) \\
&= \frac{\sigma'_{\tau_{j-1,n}}\nu_{\tau_{j-1,n}}}{2\,\Gamma_{\tau_{j-1,n}}}\mathbb{E}_{\tau_{j-1,n}}\left[\int_{0}^{\Delta_{j,n}}\left(U_{j,n}(t+\tau_{j-1,n})^2 - t\right)dt\right] + o_p\left(\frac{1}{n^2}\right) \\
&= \frac{\sigma'_{\tau_{j-1,n}}\nu_{\tau_{j-1,n}}}{2\,\Gamma_{\tau_{j-1,n}}}\mathbb{E}_{\tau_{j-1,n}}\left[\int_{0}^{\Delta^{(0)}_{j,n}}\left(U_{j,n}(t+\tau_{j-1,n})^2 - t\right)dt\right] + o_p\left(\frac{1}{n^2}\right). \qquad (131)
\end{aligned}$$

By Eq. (121), Eq. (127) and Eq. (128) and the change of variables $s = n\Gamma_{\tau_{j-1,n}}t$,

$$\mathbb{E}_{\tau_{j-1,n}}\left[\int_{0}^{\Delta^{(0)}_{j,n}}\left(U_{j,n}(\tau_{j-1,n}+t)^2 - t\right)dt\right] = \frac{1}{n^2\,\Gamma^2_{\tau_{j-1,n}}}\mathbb{E}\left[\int_0^{\tau_B}\left(B_s^2 - s\right)ds\right] = -\frac{2\,a^2\,b^2}{3\,n^2\,\Gamma^2_{\tau_{j-1,n}}}, \qquad (132)$$

where we have used Eq. (105) for $\mathbb{E}[\int_0^{\tau_B}(B_s^2 - s)\,ds]$. Plugging Eq. (132) into Eq. (131) yields Eq. (108). A similar reasoning[22] leads to Eq. (109). Next, to derive Eq. (110), start from Eq. (129) obtaining

$$\int_{\tau_{j-1,n}}^{t}\left(Z_s - Z_{\tau_{j-1,n}}\right)dW_s = \frac{\sigma'_{\tau_{j-1,n}}}{\Gamma^{1/2}_{\tau_{j-1,n}}}\int_{\tau_{j-1,n}}^{t}\left(Z_s - Z_{\tau_{j-1,n}}\right)dU_{j,n}(s) + \frac{\nu_{\tau_{j-1,n}}}{\Gamma^{1/2}_{\tau_{j-1,n}}}\int_{\tau_{j-1,n}}^{t}\left(Z_s - Z_{\tau_{j-1,n}}\right)dR_{j,n}(s).$$

The post-$\tau_{j-1,n}$ increment process $\{Z_{\tau_{j-1,n}+s} - Z_{\tau_{j-1,n}}\}_{s\geq 0}$ is independent of $\mathcal{G}_{j,n}$ and of the post-$\tau_{j-1,n}$ rotated Brownian processes. Consequently,

$$\mathbb{E}\left[\int_{\tau_{j-1,n}}^{t}\left(Z_s - Z_{\tau_{j-1,n}}\right)dU_{j,n}(s)\,\middle|\,\mathcal{G}_{j,n}\right] = 0, \quad \mathbb{E}\left[\int_{\tau_{j-1,n}}^{t}\left(Z_s - Z_{\tau_{j-1,n}}\right)dR_{j,n}(s)\,\middle|\,\mathcal{G}_{j,n}\right] = 0.$$

Hence,

$$\mathbb{E}_{\tau_{j-1,n}}\left[\int_{\tau_{j-1,n}}^{\tau_{j-1,n}+\Delta^{(0)}_{j,n}}\left(\int_{\tau_{j-1,n}}^{t}\left(Z_s - Z_{\tau_{j-1,n}}\right)dW_s\right)dt\right] = 0.$$

[22] Start from Eq. (129). For $t \in [\tau_{j-1,n}, \tau_{j,n}]$, we have

$$\begin{aligned}
\int_{\tau_{j-1,n}}^{t}\left(W_s - W_{\tau_{j-1,n}}\right)dW_s &= \frac{1}{\Gamma_{\tau_{j-1,n}}}\int_{\tau_{j-1,n}}^{t}\left(\sigma'_{\tau_{j-1,n}}U_{j,n}(s) + \nu_{\tau_{j-1,n}}R_{j,n}(s)\right)\left(\sigma'_{\tau_{j-1,n}}dU_{j,n}(s) + \nu_{\tau_{j-1,n}}dR_{j,n}(s)\right) \\
&= \frac{\sigma'^2_{\tau_{j-1,n}}}{\Gamma_{\tau_{j-1,n}}}\int_{\tau_{j-1,n}}^{t}U_{j,n}(s)\,dU_{j,n}(s) + \frac{\sigma'_{\tau_{j-1,n}}\nu_{\tau_{j-1,n}}}{\Gamma_{\tau_{j-1,n}}}\int_{\tau_{j-1,n}}^{t}U_{j,n}(s)\,dR_{j,n}(s) \\
&\quad + \frac{\sigma'_{\tau_{j-1,n}}\nu_{\tau_{j-1,n}}}{\Gamma_{\tau_{j-1,n}}}\int_{\tau_{j-1,n}}^{t}R_{j,n}(s)\,dU_{j,n}(s) + \frac{\nu^2_{\tau_{j-1,n}}}{\Gamma_{\tau_{j-1,n}}}\int_{\tau_{j-1,n}}^{t}R_{j,n}(s)\,dR_{j,n}(s).
\end{aligned}$$

Using the the approximation in Eq. (126), conditioning on $\mathcal{G}_{j,n}$, and using Itô's formula, yields

$$\mathbb{E}_{\tau_{j-1,n}}\left[\int_{\tau_{j-1,n}}^{\tau_{j,n}}\left(\int_{\tau_{j-1,n}}^{t}\left(W_s - W_{\tau_{j-1,n}}\right)dW_s\right)dt\right] = \frac{\sigma'^2_{\tau_{j-1,n}}}{\Gamma_{\tau_{j-1,n}}}\mathbb{E}_{\tau_{j-1,n}}\left[\int_0^{\Delta^{(0)}_{j,n}}\frac{1}{2}\left(U_{j,n}(\tau_{j-1,n}+t)^2 - t\right)dt\right] + o_p\left(n^{-2}\right).$$

Replicating the same steps used for Eq. (132) gives Eq. (109), as claimed.

Using Eq. (126), we conclude

$$\mathbb{E}_{\tau_{j-1,n}}\left[\int_{\tau_{j-1,n}}^{\tau_{j,n}}\left(\int_{\tau_{j-1,n}}^{t}\left(Z_s - Z_{\tau_{j-1,n}}\right)\,dW_s\right)dt\right] = o_p(n^{-2}),$$

as claimed in Eq. (110). Moving to Eq. (111), since we only need the leading $n^{-3/2}$ order, we first notice that, under the (simplifying) assumptions in Eq. (22), we have

$$\Phi_{j,n}^{(\Delta)}(t) = \mu'_{\tau_{j-1,n}}\,u_y + O_p\left(n^{-1}\right).$$

We, thus, obtain

$$\begin{aligned}
\mathbb{E}_{\tau_{j-1,n}}\left[\int_{\tau_{j-1,n}}^{\tau_{j,n}}\Phi_{j,n}^{(\Delta)}(s)\,ds\right] &= \mu'_{\tau_{j-1,n}}\,\mathbb{E}_{\tau_{j-1,n}}\left[\int_{\tau_{j-1,n}}^{\tau_{j,n}}\frac{U_n-\ell_n-2\,\delta_j^n\sigma(s)}{\Gamma_s}\,ds\right]+O_p\left(n^{-2}\right)\\
&= \mu'_{\tau_{j-1,n}}\,\frac{U_n-\ell_n}{\Gamma_{\tau_{j-1,n}}}\,\mathbb{E}_{\tau_{j-1,n}}\left[\Delta_{j,n}\right]-\frac{2\,\mu'_{\tau_{j-1,n}}}{\Gamma_{\tau_{j-1,n}}}\,\mathbb{E}_{\tau_{j-1,n}}\left[\int_{\tau_{j-1,n}}^{\tau_{j,n}}\delta_j^n\sigma(s)\,ds\right]+O_p\left(n^{-2}\right).\\
&= \mu'_{\tau_{j-1,n}}\,\frac{a\,b\,(a-b)}{n^{3/2}\,\Gamma^2_{\tau_{j-1,n}}}-\frac{2\,\mu'_{\tau_{j-1,n}}}{\Gamma_{\tau_{j-1,n}}}\,\mathbb{E}_{\tau_{j-1,n}}\left[\int_{\tau_{j-1,n}}^{\tau_{j,n}}\delta_j^n\sigma(s)\,ds\right]+O_p\left(n^{-2}\right). \qquad (133)
\end{aligned}$$

Since $\delta_j^n\sigma(s) = \sigma'_{\tau_{j-1,n}}(W_s - W_{\tau_{j-1,n}}) + \nu_{\tau_{j-1,n}}(V_s - V_{\tau_{j-1,n}}) + O_p(n^{-1})$, we can write $\delta_j^n\sigma(s) = \Gamma^{1/2}_{\tau_{j-1,n}}\,U_{j,n}(s) + O_p(n^{-1})$. Hence, using the identity in Eq. (106), we obtain

$$\mathbb{E}_{\tau_{j-1,n}}\left[\int_{\tau_{j-1,n}}^{\tau_{j,n}}\delta_j^n\sigma(s)\,ds\right] = \frac{U_n\,\ell_n\,(U_n-\ell_n)}{3\,\Gamma_{\tau_{j-1,n}}}+O_p\left(n^{-2}\right),$$

which, when plugged back into Eq. (133), delivers Eq. (111), as requested. In order to prove the expansion in Eq. (112), we follow similar steps. In particular, we start from the analogous of Eq. (129) and define

$$\widehat{W}_t^{(\|)} \doteq \int_0^t a_s\,dW_s^{(\star)} + \int_0^t b_s\,dV_s^{(\star)}, \quad \widehat{W}_t^{(\perp)} \doteq \int_0^t b_s\,dW_s^{(\star)} - \int_0^t a_s\,dV_s^{(\star)}, \qquad (134)$$

where

$$a_s \doteq \frac{\sigma'_{\tau_{j-1,n}+s}}{\sqrt{\Gamma_{\tau_{j-1,n}+s}}}, \quad b_s \doteq \frac{\nu_{\tau_{j-1,n}+s}}{\sqrt{\Gamma_{\tau_{j-1,n}+s}}}, \quad W_s^{(\star)} \doteq W_{\tau_{j-1,n}+s} - W_{\tau_{j-1,n}}, \quad V_s^{(\star)} \doteq V_{\tau_{j-1,n}+s} - V_{\tau_{j-1,n}}. \qquad (135)$$

It is immediate to see that $[\widehat{W}^{(\|)}]_s = [\widehat{W}^{(\perp)}]_s = s$ and $[\widehat{W}^{(\|)}, \widehat{W}^{(\perp)}]_s = 0$. Also, from Eq. (134), we obtain

$$dW_s^{(\star)} = a_s\,d\widehat{W}_s^{(\|)} + b_s\,d\widehat{W}_s^{(\perp)}, \quad dV_s^{(\star)} = b_s\,d\widehat{W}_s^{(\|)} - a_s\,d\widehat{W}_s^{(\perp)}. \qquad (136)$$

In particular, after defining $\sigma_s^{(\star)} \doteq \sigma_{\tau_{j-1,n}+s} - \sigma_{\tau_{j-1,n}}$, we have

$$d\sigma_s^{(\star)} = \mu'_{\tau_{j-1,n}+s}\,ds + \sqrt{\Gamma_{\tau_{j-1,n}+s}}\,d\widehat{W}_s^{(\|)}. \qquad (137)$$

The time duration is now represented as (notice that $\sigma_0^{(\star)} = 0$)

$$\Delta_{j,n} = \inf\left\{s \geq 0 \;\middle|\; \sigma_s^{(\star)} \notin \left(-\frac{b}{\sqrt{n}}, +\frac{a}{\sqrt{n}}\right)\right\}. \qquad (138)$$

We, now, have that

$$\mathbb{E}_{\tau_{j-1,n}}\left[\int_{\tau_{j-1,n}}^{\tau_{j-1,n}+\Delta_{j,n}}\left(W_s - W_{\tau_{j-1,n}}\right)\,ds\right] = \mathbb{E}_{\tau_{j-1,n}}\left[\int_0^{\Delta_{j,n}} W_s^{(\star)}\,ds\right] = \mathbb{E}_{\tau_{j-1,n}}\left[\int_0^{\Delta_{j,n}} M_s^{(\|)}\,ds + \int_0^{\Delta_{j,n}} M_s^{(\perp)}\,ds\right],$$

where

$$M_s^{(\|)} \doteq \int_0^s a_u\,d\widehat{W}_u^{(\|)}, \quad M_s^{(\perp)} \doteq \int_0^s b_u\,d\widehat{W}_u^{(\perp)},$$

In the case of the (simplifying) assumptions in Eq. (22), we have that σ' and ν are independent of W and V. Thus, conditionally on σ' and ν, the process $\widehat{W}^{(\perp)}$ in Eq. (134) is Brownian and the process b in Eq. (135) is deterministic, whence

$$\mathbb{E}\left[M_s^{(\perp)} \,\middle|\, \mathcal{F}_{\tau_{j-1,n}} \vee \sigma\left\{\left(\sigma'_{\tau_{j-1,n}+u}, \nu_{\tau_{j-1,n}+u}\right)_{u\geq 0}\right\}\right] = 0.$$

Moreover, since conditionally on σ' and ν, $\widehat{W}^{(\parallel)}$ and $\widehat{W}^{(\perp)}$ are independent of each other, we have

$$\mathbb{E}\left[M_s^{(\perp)} \,\middle|\, \mathcal{F}_{\tau_{j-1,n}} \vee \sigma\left\{\left(\sigma'_{\tau_{j-1,n}+u}, \nu_{\tau_{j-1,n}+u}, \widehat{W}_u^{(\parallel)}\right)_{u\geq 0}\right\}\right] = 0.$$

Next, notice that the time interval $\Delta_{j,n}$ in Eq. (138) is measurable with respect to

$$\mathcal{C} \doteq \mathcal{F}_{\tau_{j-1,n}} \vee \sigma\left\{\left(\mu'_{\tau_{j-1,n}+u}, \sigma'_{\tau_{j-1,n}+u}, \nu_{\tau_{j-1,n}+u}, \widehat{W}_u^{(\parallel)}\right)_{u\geq 0}\right\},$$

and so

$$\mathbb{E}_{\tau_{j-1,n}}\left[\int_0^{\Delta_{j,n}} M_s^{(\perp)}\, ds\right] = \mathbb{E}_{\tau_{j-1,n}}\left[\mathbb{E}\left[\int_0^{\Delta_{j,n}} M_s^{(\perp)}\, ds \,\middle|\, \mathcal{C}\right]\right] = \mathbb{E}_{\tau_{j-1,n}}\left[\int_0^{\Delta_{j,n}} \mathbb{E}\left[M_s^{(\perp)} \,\middle|\, \mathcal{C}\right] ds\right] = 0,$$

where we have also used the fact that $M^{(\perp)}$ is independent of μ'. Accordingly,

$$\mathbb{E}_{\tau_{j-1,n}}\left[\int_{\tau_{j-1,n}}^{\tau_{j-1,n}+\Delta_{j,n}} \left(W_s - W_{\tau_{j-1,n}}\right) ds\right] = \mathbb{E}_{\tau_{j-1,n}}\left[\int_0^{\Delta_{j,n}} M_s^{(\parallel)}\, ds\right].$$

It is now enough to compute an expansion for $\mathbb{E}_{\tau_{j-1,n}}[\int_0^{\Delta_{j,n}} M_s^{(\parallel)}\, ds]$. To obtain the expansion up to order n^{-2}, consider the process

$$d\widetilde{\sigma}_s^{(\star)} = \mu'_{\tau_{j-1,n}}\, ds + \sqrt{\Gamma_{\tau_{j-1,n}+s}}\, d\widehat{W}_s^{(\parallel)},$$

which is obtained from the process $\sigma_s^{(\star)}$ in Eq. (137) by freezing the drift coefficient. Under the assumption that μ' is a continuous semimartingale, it can be easily shown that

$$\mathbb{E}_{\tau_{j-1,n}}\left[\int_0^{\Delta_{j,n}} M_s^{(\parallel)}\, ds\right] = \mathbb{E}_{\tau_{j-1,n}}\left[\int_0^{\widetilde{\Delta}_{j,n}} M_s^{(\parallel)}\, ds\right] + O_p\left(n^{-5/2}\right), \tag{139}$$

where $\widetilde{\Delta}_{j,n}$ is defined as

$$\widetilde{\Delta}_{j,n} = \inf\left\{s \geq 0 \,\middle|\, \widetilde{\sigma}_s^{(\star)} \notin \left(-\frac{b}{\sqrt{n}}, +\frac{a}{\sqrt{n}}\right)\right\}. \tag{140}$$

Now, an application of Dynkin's formula gives

$$\mathbb{E}_{\tau_{j-1,n}}\left[\int_0^{\widetilde{\Delta}_{j,n}} M_s^{(\parallel)}\, ds\right] = \frac{a\, b\, (a-b)\, \sigma'_{\tau_{j-1,n}}}{3\, n^{3/2} \Gamma^2_{\tau_{j-1,n}}} - \frac{2\, a^2\, b^2\, \sigma'_{\tau_{j-1,n}}\, \mu'_{\tau_{j-1,n}}}{3\, n^2\, \Gamma^3_{\tau_{j-1,n}}} + O_p\left(n^{-5/2}\right),$$

which combined with Eq. (139) gives Eq. (112). Similar computations prove Eq. (113) and Eq. (114). □

Corollary A.3. *Following the logic of Lemma 12, we have*

$$\mathbb{E}_{\tau_{j-1,n}}\left[\int_{\tau_{j-1,n}}^{\tau_{j,n}} \left(\int_{\tau_{j-1,n}}^{t} \left(W_s - W_{\tau_{j-1,n}}\right) dV_s\right) dt\right] = -\frac{a^2 b^2}{3} \frac{\sigma'_{\tau_{j-1,n}} \nu_{\tau_{j-1,n}}}{\Gamma^3_{\tau_{j-1,n}}} \frac{1}{n^2} + o_p(n^{-2}), \tag{141}$$

$$\mathbb{E}_{\tau_{j-1,n}}\left[\int_{\tau_{j-1,n}}^{\tau_{j,n}} \left(\int_{\tau_{j-1,n}}^{t} \left(V_s - V_{\tau_{j-1,n}}\right) dV_s\right) dt\right] = -\frac{a^2 b^2}{3} \frac{\nu^2_{\tau_{j-1,n}}}{\Gamma^3_{\tau_{j-1,n}}} \frac{1}{n^2} + o_p(n^{-2}) \tag{142}$$

$$\mathbb{E}_{\tau_{j-1,n}}\left[\int_{\tau_{j-1,n}}^{\tau_{j,n}} \left(\int_{\tau_{j-1,n}}^{t} \left(B_s - B_{\tau_{j-1,n}}\right) dV_s\right) dt\right] = o_p(n^{-2}). \tag{143}$$

Lemma 13. *Let Assumptions 1, 2, 6, and 7 hold. Then, for any bounded càdlàg process G, we obtain*

$$\frac{1}{n}\sum_{\tau_{j,n}\le t} G_{\tau_{j-1,n}} \xrightarrow{p} \frac{1}{a\,b}\int_0^t G_s\,\Gamma_s\,ds.$$

In particular,

$$\frac{1}{n}\sum_{\tau_{j,n}\le t} = \frac{N_{t,n}}{n} \xrightarrow{p} \frac{1}{a\,b}\int_0^t \Gamma_s\,ds.$$

Proof. The result is proved by simply adapting the proof of Lemma 10. □

Proof of Theorem 3.4. Combining the expansion in Eq. (112) of Lemma 12 with the identity in Eq. (44) of Lemma 6 (which is valid for any sampling scheme), we obtain

$$\begin{aligned}
\mathbb{E}_{\tau_{j-1,n}}\left[\Delta_{j,n}\left(\Delta_j^n W\right)\right] &= \frac{a\,b\,(a-b)}{3\,n^{3/2}\,\Gamma^2_{\tau_{j-1,n}}}\,\sigma'_{\tau_{j-1,n}} + O_p\left(n^{-2}\right), \qquad (144)\\
\mathbb{E}_{\tau_{j-1,n}}\left[\Delta_{j,n}\left(\Delta_j^n V\right)\right] &= \frac{a\,b\,(a-b)}{3\,n^{3/2}\,\Gamma^2_{\tau_{j-1,n}}}\,\nu_{\tau_{j-1,n}} + O_p\left(n^{-2}\right).
\end{aligned}$$

The bias of the tricity is now derived starting, as usual, from the approximation $\Delta_j^n X = \sigma_{\tau_{j-1,n}}\,\Delta_j^n W + O_p(n^{-1})$ which, using the third Bartlett identity, delivers the required convergence:

$$\begin{aligned}
\sum_{\tau_{j,n}\le t}\sqrt{n}\,\mathbb{E}_{\tau_{j-1,n}}\left[\left(\Delta_j^n X\right)^3\right] &= 3\sum_{\tau_{j,n}\le t}\sqrt{n}\,\sigma^3_{\tau_{j-1,n}}\mathbb{E}_{\tau_{j-1,n}}\left[\Delta_{j,n}\left(\Delta_j^n W\right)\right] + O_p\left(n^{-1/2}\right)\\
&= \frac{a\,b\,(a-b)}{n}\sum_{\tau_{j,n}\le t}\frac{\sigma^3_{\tau_{j-1,n}}}{\Gamma^2_{\tau_{j-1,n}}}\,\sigma'_{\tau_{j-1,n}} + O_p\left(n^{-1/2}\right) \xrightarrow{p} (a-b)\int_0^t \frac{\sigma_s^3}{\Gamma_s}\,\sigma'_s\,ds, \qquad (145)
\end{aligned}$$

where we have used Eq. (144) and Lemma 13.

To proceed with the remaining part of the statement, call $\varphi_s \doteq (a-b)\,\sigma_s^3\,\sigma'_s/\Gamma_s$. For the stable central limit theorem, consider the estimation error decomposition

$$\begin{aligned}
&\sqrt{n}\left(\sum_{\tau_{j,n}\le t}\sqrt{n}\left(\Delta_j^n X\right)^3 - \int_0^t \varphi_s\,ds\right) =\\
&= \sqrt{n}\sum_{\tau_{j,n}\le t}\left(\sqrt{n}\left(\Delta_j^n X\right)^3 - \varphi_{\tau_{j-1,n}}\,\Delta_{j,n}\right) - \sqrt{n}\sum_{\tau_{j,n}\le t}\int_{\tau_{j-1,n}}^{\tau_{j,n}}\left(\varphi_s - \varphi_{\tau_{j-1,n}}\right)ds - \sqrt{n}\int_{\tau_{N_{t,n},n}}^t \varphi_s\,ds\\
&= \sum_{\tau_{j,n}\le t}\xi_{j,n} + \sum_{\tau_{j,n}\le t}\xi'_{j,n} - \sqrt{n}\int_{\tau_{N_{t,n},n}}^t \varphi_s\,ds,
\end{aligned}$$

where the triangular arrays $\{\xi_{j,n}\}$ and $\{\xi'_{j,n}\}$ are defined as

$$\begin{aligned}
\xi_{j,n} &\doteq \sqrt{n}\left(\sqrt{n}\left(\Delta_j^n X\right)^3 - \varphi_{\tau_{j-1,n}}\,\Delta_{j,n}\right),\\
\xi'_{j,n} &\doteq -\sqrt{n}\int_{\tau_{j-1,n}}^{\tau_{j,n}}\left(\varphi_s - \varphi_{\tau_{j-1,n}}\right)ds.
\end{aligned}$$

We start from the approximation in Eq. (98), which we report below again for clarity:

$$\mathbb{E}_{\tau_{j-1,n}}\left[\left(\Delta_j^n X\right)^3\right] = \mathbb{E}_{\tau_{j-1,n}}\left[\left(\int_{\tau_{j-1,n}}^{\tau_{j,n}}\sigma_s\,dW_s\right)^3\right] + 3\,\mathbb{E}_{\tau_{j-1,n}}\left[\left(\int_{\tau_{j-1,n}}^{\tau_{j,n}}\mu_s\,ds\right)\left(\int_{\tau_{j-1,n}}^{\tau_{j,n}}\sigma_s\,dW_s\right)^2\right] + O_p\left(n^{-5/2}\right).$$

Using $\int_{\tau_{j-1,n}}^{\tau_{j,n}} \sigma_s\, dW_s = \sigma_{\tau_{j-1,n}}\, \Delta_j^n W + \int_{\tau_{j-1,n}}^{\tau_{j,n}} (\sigma_s - \sigma_{\tau_{j-1,n}})\, dW_s$ and a similar decomposition for $\int_{\tau_{j-1,n}}^{\tau_{j,n}} \mu_s\, ds$, we obtain

$$\begin{aligned}
\mathbb{E}_{\tau_{j-1,n}}\left[\left(\Delta_j^n X\right)^3\right] &= \sigma^3_{\tau_{j-1,n}}\, \mathbb{E}_{\tau_{j-1,n}}\left[\left(\Delta_j^n W\right)^3\right] \\
&\quad + 3\,\sigma^2_{\tau_{j-1,n}}\, \mathbb{E}_{\tau_{j-1,n}}\left[\left(\Delta_j^n W\right)^2 \int_{\tau_{j-1,n}}^{\tau_{j,n}} \left(\sigma_s - \sigma_{\tau_{j-1,n}}\right) dW_s\right] \\
&\quad + 3\,\mu_{\tau_{j-1,n}}\, \sigma^2_{\tau_{j-1,n}}\, \mathbb{E}_{\tau_{j-1,n}}\left[\Delta_{j,n}\left(\Delta_j^n W\right)^2\right] + O_p\left(n^{-5/2}\right),
\end{aligned}$$

from which, using the expansions of Lemma 12 (in one case combined with the identity in Eq. (44) of Lemma 6), we have

$$\begin{aligned}
\mathbb{E}_{\tau_{j-1,n}}\left[\xi_{j,n}\right] &= 3\,n\,\sigma^3_{\tau_{j-1,n}}\, \mathbb{E}_{\tau_{j-1,n}}\left[\Delta_{j,n}\left(\Delta_j^n W\right)\right] - \sqrt{n}\,\varphi_{\tau_{j-1,n}}\, \mathbb{E}_{\tau_{j-1,n}}\left[\Delta_{j,n}\right] \\
&\quad + 3\,n\,\sigma^2_{\tau_{j-1,n}}\, \mathbb{E}_{\tau_{j-1,n}}\left[\left(\Delta_j^n W\right)^2 \int_{\tau_{j-1,n}}^{\tau_{j,n}} \left(\sigma_s - \sigma_{\tau_{j-1,n}}\right) dW_s\right] \\
&\quad + 3\,n\,\mu_{\tau_{j-1,n}}\, \sigma^2_{\tau_{j-1,n}}\, \mathbb{E}_{\tau_{j-1,n}}\left[\Delta_{j,n}\left(\Delta_j^n W\right)^2\right] + O_p\left(n^{-3/2}\right) \\
&= 3\,n\,\sigma^3_{\tau_{j-1,n}}\left(\sigma'_{\tau_{j-1,n}}\left(\frac{ab(a-b)}{3\,\Gamma^2_{\tau_{j-1,n}}\,n^{3/2}} - \frac{2a^2b^2}{3}\,\frac{\mu'_{\tau_{j-1,n}}}{\Gamma^3_{\tau_{j-1,n}}\,n^2}\right)\right) \\
&\quad - \sqrt{n}\,(a-b)\,\frac{\sigma^3_{\tau_{j-1,n}}\,\sigma'_{\tau_{j-1,n}}}{\Gamma_{\tau_{j-1,n}}}\left(\frac{a\,b}{n\,\Gamma_{\tau_{j-1,n}}} + \frac{\mu'_{\tau_{j-1,n}}\,a\,b\,(a-b)}{3\,n^{3/2}\,\Gamma^2_{\tau_{j-1,n}}}\right) \\
&\quad + 3\,n\,\sigma^2_{\tau_{j-1,n}}\,\frac{ab\,(a-b)^2}{3}\,\frac{\sigma'_{\tau_{j-1,n}}}{n^2\,\Gamma^2_{\tau_{j-1,n}}} \\
&\quad + 3\,n\,\mu_{\tau_{j-1,n}}\,\sigma^2_{\tau_{j-1,n}}\left(\frac{ab(a^2+3ab+b^2)}{3\,n^2\Gamma^2_{\tau_{j-1,n}}} - \frac{2a^2b^2}{3}\,\frac{\sigma'^2_{\tau_{j-1,n}}}{n^2\Gamma^3_{\tau_{j-1,n}}}\right) + O_p\left(n^{-3/2}\right) \\
&= -\mu'_{\tau_{j-1,n}}\,\sigma^3_{\tau_{j-1,n}}\,\sigma'_{\tau_{j-1,n}}\,\frac{2a^2b^2}{n\,\Gamma^3_{\tau_{j-1,n}}} - \mu'_{\tau_{j-1,n}}\,\sigma^3_{\tau_{j-1,n}}\,\sigma'_{\tau_{j-1,n}}\,\frac{a\,b\,(a-b)^2}{3\,n\,\Gamma^3_{\tau_{j-1,n}}} \\
&\quad + \sigma^2_{\tau_{j-1,n}}\,\sigma'_{\tau_{j-1,n}}\,\frac{ab\,(a-b)^2}{n\,\Gamma^2_{\tau_{j-1,n}}} + \mu_{\tau_{j-1,n}}\,\sigma^2_{\tau_{j-1,n}}\,\frac{ab(a^2+3ab+b^2)}{n\,\Gamma^2_{\tau_{j-1,n}}} \\
&\quad - \mu_{\tau_{j-1,n}}\,\sigma^2_{\tau_{j-1,n}}\,\sigma'^2_{\tau_{j-1,n}}\,\frac{2a^2b^2}{n\,\Gamma^3_{\tau_{j-1,n}}} + O_p\left(n^{-3/2}\right).
\end{aligned} \tag{146}$$

Lemma 13 provides the convergence of the Riemann sum:

$$\begin{aligned}
\sum_{\tau_{j,n}\le t} \mathbb{E}_{\tau_{j-1,n}}\left[\xi_{j,n}\right] \xrightarrow{p} &-\frac{a^2+4\,a\,b+b^2}{3}\int_0^t \frac{\mu'_s\,\sigma^3_s\,\sigma'_s}{\Gamma^2_s}\,ds + (a-b)^2\int_0^t \frac{\sigma^2_s\,\sigma'_s}{\Gamma_s}\,ds \\
&+ (a^2+3ab+b^2)\int_0^t \frac{\mu_s\,\sigma^2_s}{\Gamma_s}\,ds - 2a\,b\int_0^t \frac{\mu_s\,\sigma^2_s\,\sigma'^2_s}{\Gamma^2_s}\,ds.
\end{aligned}$$

Similarly, using standard Itô calculus, we obtain

$$\sum_{\tau_{j,n}\le t} \mathbb{E}_{\tau_{j-1,n}}\left[\xi'_{j,n}\right] \xrightarrow{p} -(a-b)^2\int_0^t \frac{\sigma^2_s\,\sigma'_s}{\Gamma_s}\,ds.$$

Summing up, we derive the finite-variation process defined in Eq. (24) of the statement:

$$\begin{aligned}\sum_{\tau_{j,n}\le t}\mathbb{E}_{\tau_{j-1,n}}\left[\xi_{j,n}+\xi'_{j,n}\right] &\xrightarrow{p}\mathcal{B}_t\\ &=-\frac{a^2+4\,a\,b+b^2}{3}\int_0^t\frac{\mu'_s\,\sigma_s^3\,\sigma'_s}{\Gamma_s^2}\,ds+(a^2+3ab+b^2)\int_0^t\frac{\mu_s\,\sigma_s^2}{\Gamma_s}\,ds\\ &\quad-2a\,b\int_0^t\frac{\mu_s\,\sigma_s^2\,\sigma_s'^2}{\Gamma_s^2}\,ds.\end{aligned}$$

To compute the total asymptotic variance, we first notice, from previous computations, that

$$\mathbb{E}_{\tau_{j-1,n}}\left[\xi_{j,n}\right]=O_p\left(n^{-1}\right),\quad \mathbb{E}_{\tau_{j-1,n}}\left[\xi'_{j,n}\right]=O_p\left(n^{-1}\right),$$

whence

$$\sum_{\tau_{j,n}\le t}\left(\mathbb{E}_{\tau_{j-1,n}}\left[\left(\xi_{j,n}+\xi'_{j,n}\right)^2\right]-\left(\mathbb{E}_{\tau_{j-1,n}}\left[\xi_{j,n}+\xi'_{j,n}\right]\right)^2\right)=\sum_{\tau_{j,n}\le t}\mathbb{E}_{\tau_{j-1,n}}\left[\left(\xi_{j,n}+\xi'_{j,n}\right)^2\right]+o_p(1).$$

Moreover, since

$$\xi_{j,n}=\sqrt{n}\left(\sqrt{n}\left(\Delta_j^n X\right)^3-\varphi_{\tau_{j-1,n}}\Delta_{j,n}\right)=\underbrace{n\,\sigma^3_{\tau_{j-1,n}}\left(\Delta_j^n W\right)^3-\sqrt{n}\,\varphi_{\tau_{j-1,n}}\Delta_{j,n}}_{\eta_{j,n}}+r_{j,n}\tag{147}$$

with $r_{j,n}=O_p(n^{-1})$, we obtain

$$\begin{aligned}\sum_{\tau_{j,n}\le t}\mathbb{E}_{\tau_{j-1,n}}\left[\left(\xi_{j,n}+\xi'_{j,n}\right)^2\right] &= \sum_{\tau_{j,n}\le t}\left(\mathbb{E}_{\tau_{j-1,n}}\left[\left(\eta_{j,n}+\xi'_{j,n}\right)^2\right]+\mathbb{E}_{\tau_{j-1,n}}\left[r_{j,n}^2\right]+2\,\mathbb{E}_{\tau_{j-1,n}}\left[r_{j,n}\left(\eta_{j,n}+\xi'_{j,n}\right)\right]\right)\\ &= \sum_{\tau_{j,n}\le t}\mathbb{E}_{\tau_{j-1,n}}\left[\left(\eta_{j,n}+\xi'_{j,n}\right)^2\right]+o_p(1)\\ &= \sum_{\tau_{j,n}\le t}\mathbb{E}_{\tau_{j-1,n}}\left[\left(\eta_{j,n}^2+\xi_{j,n}'^2+2\,\eta_{j,n}\,\xi'_{j,n}\right)\right]+o_p(1)\\ &= \sum_{\tau_{j,n}\le t}\mathbb{E}_{\tau_{j-1,n}}\left[\eta_{j,n}^2\right]+o_p(1),\end{aligned}\tag{148}$$

where, in the second equation, we have used

$$r_{j,n}^2=O_p\left(n^{-2}\right),\quad r_{j,n}\,\eta_{j,n}=O_p\left(n^{-3/2}\right),\quad r_{j,n}\,\xi'_{j,n}=O_p\left(n^{-2}\right),$$

while, in the last equation, we have used

$$\xi_{j,n}'^2=O_p\left(n^{-2}\right),\quad \eta_{j,n}\,\xi'_{j,n}=O_p\left(n^{-3/2}\right).$$

To compute the total asymptotic variance, it is now enough to compute the limit of

$$\sum_{\tau_{j,n}\le t}\mathbb{E}_{\tau_{j-1,n}}\left[\eta_{j,n}^2\right].$$

For this purpose, we follow a similar logic used in the definition of the Brownian motions in Eq. (115). For simplicity, we set

$$t_{j-1}\doteq\tau_{j-1,n},\qquad \sigma_0\doteq\sigma_{t_{j-1}},\qquad \sigma'_0\doteq\sigma'_{t_{j-1}},\qquad \nu_0\doteq\nu_{t_{j-1}},\qquad \Gamma_0\doteq\sigma_0'^2+\nu_0^2.$$

We also introduce the normalized coefficients

$$\alpha\doteq\frac{\sigma'_0}{\sqrt{\Gamma_0}},\qquad \beta\doteq\frac{\nu_0}{\sqrt{\Gamma_0}},\qquad \alpha^2+\beta^2=1,\qquad d\doteq a-b.$$

Consider, now, the rotated Brownian motions

$$B_u^{(1)} \doteq \alpha\big(W_{t_{j-1}+u} - W_{t_{j-1}}\big) + \beta\big(V_{t_{j-1}+u} - V_{t_{j-1}}\big),$$
$$B_u^{(2)} \doteq -\beta\big(W_{t_{j-1}+u} - W_{t_{j-1}}\big) + \alpha\big(V_{t_{j-1}+u} - V_{t_{j-1}}\big).$$

Conditionally on $\mathcal{F}_{t_{j-1}}$, these are independent standard Brownian motions. Let τ^* be the first exit time of $B^{(1)}$ from

$$\left(-\frac{b}{\sqrt{n}\sqrt{\Gamma_0}}, \frac{a}{\sqrt{n}\sqrt{\Gamma_0}}\right).$$

Define the rescaled exit variables

$$U \doteq \sqrt{n}\sqrt{\Gamma_0}\, B_{\tau^*}^{(1)} \in \{a, -b\}, \qquad T \doteq n\Gamma_0\tau^*, \qquad C \doteq \sqrt{n}\sqrt{\Gamma_0}\, B_{\tau^*}^{(2)}.$$

Given the representation[23]

$$\Delta_j^n W = \frac{\alpha U - \beta C}{\sqrt{n}\sqrt{\Gamma_0}} + o_p\left(n^{-1/2}\right),$$

and, using the approximation in Eq. (123), we have

$$\Delta_{j,n} = \frac{T}{n\Gamma_0} + o_p\left(n^{-1}\right).$$

We obtain

$$\eta_{j,n} = n\,\sigma^3_{\tau_{j-1,n}}\left(\Delta_j^n W\right)^3 - \sqrt{n}\,\varphi_{\tau_{j-1,n}}\,\Delta_{j,n} = n\sigma_0^3\left(\frac{\alpha U - \beta C}{\sqrt{n}\sqrt{\Gamma_0}}\right)^3 - \sqrt{n}\,d\,\frac{\sigma_0^3\alpha}{\sqrt{\Gamma_0}}\cdot\frac{T}{n\Gamma_0} + o_p(n^{-1/2})$$
$$= \frac{\sigma_0^3}{\sqrt{n}\,\Gamma_0^{3/2}}\Big((\alpha U - \beta C)^3 - d\alpha T\Big) + o_p(n^{-1/2}).$$

Call $A \doteq \alpha U - \beta C$, to write

$$\mathbb{E}_{\tau_{j-1,n}}\left[\eta_{j,n}^2\right] = \frac{\sigma_0^6}{n\Gamma_0^3}\,\mathbb{E}\big[(A^3 - d\alpha T)^2 \mid \mathcal{F}_{t_{j-1}}\big] + o_p(n^{-1}). \tag{149}$$

By the usual scaling laws, we have $C \mid (\mathcal{F}_{t_{j-1}} \vee \sigma(T,U)) \sim N(0,T)$. The conditional law of (T,U,C) given $\mathcal{F}_{t_{j-1}}$ is the fixed Brownian exit law; hence moments involving only (T,U,C) can also be written without conditioning on $\mathcal{F}_{t_{j-1}}$. In particular,

$$\mathbb{E}\Big[C \,\Big|\, \mathcal{F}_{t_{j-1}} \vee \sigma(T,U)\Big] = 0, \qquad \mathbb{E}\Big[C^2 \,\Big|\, \mathcal{F}_{t_{j-1}} \vee \sigma(T,U)\Big] = T,$$
$$\mathbb{E}\Big[C^4 \,\Big|\, \mathcal{F}_{t_{j-1}} \vee \sigma(T,U)\Big] = 3T^2, \qquad \mathbb{E}\Big[C^6 \,\Big|\, \mathcal{F}_{t_{j-1}} \vee \sigma(T,U)\Big] = 15T^3.$$

Now, expand A^3 explicitly

$$A^3 = (\alpha U - \beta C)^3 = \alpha^3U^3 - 3\alpha^2\beta U^2C + 3\alpha\beta^2UC^2 - \beta^3C^3,$$

take the conditional expectation and use the fact that odd centered Gaussian moments vanish, to derive

$$\mathbb{E}\Big[A^3 \,\Big|\, \mathcal{F}_{t_{j-1}} \vee \sigma(T,U)\Big] = \alpha^3U^3 + 3\alpha\beta^2UT. \tag{150}$$

Similarly,

$$\mathbb{E}\Big[A^6 \,\Big|\, \mathcal{F}_{t_{j-1}} \vee \sigma(T,U)\Big] = \alpha^6U^6 + 15\alpha^4\beta^2U^4T + 45\alpha^2\beta^4U^2T^2 + 15\beta^6T^3. \tag{151}$$

Now, write

$$\mathbb{E}_{\tau_{j-1,n}}\left[(A^3 - d\alpha T)^2\right] = \mathbb{E}_{\tau_{j-1,n}}\left[A^6\right] - 2d\alpha\,\mathbb{E}_{\tau_{j-1,n}}\left[TA^3\right] + d^2\alpha^2\mathbb{E}\left[T^2\right].$$

[23] Note that $\Delta_j^n W = W_{\tau_{j,n}} - W_{\tau_{j-1,n}}$ while U and C are sampled at $\tau^\star$ by freezing the coefficients at the beginning of the interval, which gives the $o_p(n^{-1/2})$ in the representation.

From Eq. (150),

$$\mathbb{E}_{\tau_{j-1,n}}\left[TA^3\right] = \mathbb{E}_{\tau_{j-1,n}}\left[T\,\mathbb{E}\left[A^3 \,\middle|\, \mathcal{F}_{t_{j-1}} \vee \sigma(T,U)\right]\right] = \mathbb{E}_{\tau_{j-1,n}}\left[T(\alpha^3U^3 + 3\alpha\beta^2UT)\right] = \alpha^3\mathbb{E}\left[TU^3\right] + 3\alpha\beta^2\mathbb{E}\left[T^2U\right].$$

Using, also, Eq. (151), we conclude that

$$\begin{aligned}\mathbb{E}_{\tau_{j-1,n}}\left[(A^3 - d\alpha T)^2\right] = \alpha^6\mathbb{E}\left[U^6\right] + 15\alpha^4\beta^2\mathbb{E}\left[TU^4\right] + 45\alpha^2\beta^4\mathbb{E}\left[T^2U^2\right] + 15\beta^6\mathbb{E}\left[T^3\right] \\ - 2d\alpha^4\mathbb{E}\left[TU^3\right] - 6d\alpha^2\beta^2\mathbb{E}\left[T^2U\right] + d^2\alpha^2\mathbb{E}\left[T^2\right].\end{aligned} \tag{152}$$

Thus, everything reduces to the exit-law moments of (T,U). Here, $U \in \{a,-b\}$ is the exit location of a standard Brownian motion from $(-b,a)$ and T is the corresponding exit time. From Lemmas 6 and 11, we have

$$\mathbb{E}\left[T\right] = ab, \qquad \mathbb{E}\left[TU\right] = \frac{ab(a-b)}{3}, \qquad \mathbb{E}\left[T^2\right] = \frac{ab(a^2+3ab+b^2)}{3}. \tag{153}$$

We now compute the additional moments needed in Eq. (152). Since $U \in \{a,-b\}$ and $\mathbb{E}[U] = 0$, we have

$$\mathbb{P}\left[U = a\right] = \frac{b}{a+b}, \qquad \mathbb{P}\left[U = -b\right] = \frac{a}{a+b}.$$

Therefore,[24]

$$\mathbb{E}\left[U^4\right] = \frac{b}{a+b}a^4 + \frac{a}{a+b}b^4 = ab(a^2 - ab + b^2), \tag{154}$$

$$\mathbb{E}\left[U^5\right] = \frac{b}{a+b}a^5 + \frac{a}{a+b}(-b)^5 = ab(a-b)(a^2+b^2), \tag{155}$$

$$\mathbb{E}\left[U^6\right] = \frac{b}{a+b}a^6 + \frac{a}{a+b}b^6 = ab(a^4 - a^3b + a^2b^2 - ab^3 + b^4). \tag{156}$$

Because U takes only the two values a and $-b$, every polynomial in U is affine in U. In particular,

$$U^2 = ab + (a-b)U, \tag{157}$$

$$U^3 = ab(a-b) + (a^2 - ab + b^2)U, \tag{158}$$

$$U^4 = ab(a^2 - ab + b^2) + (a-b)(a^2+b^2)U. \tag{159}$$

Starting from Eq. (158), we arrive at $\mathbb{E}[TU^3] = ab(a-b)\mathbb{E}[T] + (a^2 - ab + b^2)\mathbb{E}[TU]$. Insert, now, Eq. (153) to obtain

$$\mathbb{E}\left[TU^3\right] = ab(a-b)\,ab + (a^2 - ab + b^2)\,\frac{ab(a-b)}{3} = ab(a-b)\left(ab + \frac{a^2 - ab + b^2}{3}\right).$$

Simplifying, we have

$$\mathbb{E}\left[TU^3\right] = \frac{ab(a-b)(a+b)^2}{3}. \tag{160}$$

From the fifth Bartlett identity, we also have

$$\mathbb{E}\left[U^5\right] - 10\mathbb{E}\left[TU^3\right] + 15\mathbb{E}\left[T^2U\right] = 0,$$

and so, accordingly,

$$\mathbb{E}\left[T^2U\right] = \frac{10\,\mathbb{E}\left[TU^3\right] - \mathbb{E}\left[U^5\right]}{15}.$$

[24] All these identities can be derived as follows. For a generic n, we have

$$\frac{b}{a+b}a^n + \frac{a}{a+b}(-b)^n = \frac{ab\left(a^{n-1} - (-b)^{n-1}\right)}{a+b}.$$

Then, use the standard formula

$$\frac{x^m - y^m}{x-y} = x^{m-1} + x^{m-2}y + \cdots + y^{m-1}.$$

With $x = a$ and $y = -b$, this becomes

$$\frac{a^m - (-b)^m}{a+b} = a^{m-1} - a^{m-2}b + a^{m-3}b^2 - \cdots + (-1)^{m-1}b^{m-1}.$$

Thus, in general,

$$\frac{b}{a+b}a^n + \frac{a}{a+b}(-b)^n = ab\left(a^{n-2} - a^{n-3}b + a^{n-4}b^2 - \cdots + (-1)^{n-2}b^{n-2}\right).$$

Using the identities in Eq. (160) and Eq. (155), with some algebra, we get

$$\mathbb{E}\left[T^2U\right] = \frac{ab(a-b)(7a^2+20ab+7b^2)}{45}. \tag{161}$$

Starting, now, from Eq. (157), we arrive at $\mathbb{E}[T^2U^2] = ab\,\mathbb{E}[T^2] + (a-b)\mathbb{E}[T^2U]$ which, inserting Eq. (153) and Eq. (161), becomes (after some algebra)

$$\mathbb{E}\left[T^2U^2\right] = \frac{ab(7a^4+21a^3b+19a^2b^2+21ab^3+7b^4)}{45}. \tag{162}$$

Starting from Eq. (159), we arrive at $\mathbb{E}[TU^4] = ab(a^2-ab+b^2)\mathbb{E}[T] + (a-b)(a^2+b^2)\mathbb{E}[TU]$ which, inserting Eq. (153) becomes (as usual, after some algebra)

$$\mathbb{E}\left[TU^4\right] = \frac{ab(a^4+a^3b-a^2b^2+ab^3+b^4)}{3}. \tag{163}$$

The expected value $\mathbb{E}[T^3]$ can be computed from the sixth Bartlett identity. In fact

$$\mathbb{E}\left[U^6\right] - 15\,\mathbb{E}\left[T\,U^4\right] + 45\,\mathbb{E}\left[T^2\,U^2\right] - 15\,\mathbb{E}\left[T^3\right] = 0 \Rightarrow \mathbb{E}\left[T^3\right] = \frac{\mathbb{E}\left[U^6\right] - 15\mathbb{E}\left[T\,U^4\right] + 45\mathbb{E}\left[T^2\,U^2\right]}{15}.$$

Insert now, in the last expression, Eq. (156), Eq. (163) and Eq. (162), to obtain (after some algebra)

$$\mathbb{E}\left[T^3\right] = \frac{ab\,(3\,a^4+15\,a^3\,b+25\,a^2\,b^2+15\,a\,b^3+3\,b^4)}{15}. \tag{164}$$

Set $u \doteq \alpha^2 = \sigma_0'^2/\Gamma_0$ and, accordingly, $1-u = \beta^2 = \nu_0^2/\Gamma_0$. Then, Eq. (152) becomes

$$\begin{aligned}\mathbb{E}_{\tau_{j-1,n}}\left[(A^3-d\alpha T)^2\right] &= u^3\mathbb{E}\left[U^6\right] + 15\,u^2\,(1-u)\,\mathbb{E}\left[T\,U^4\right] + 45\,u\,(1-u)^2\mathbb{E}\left[T^2\,U^2\right] \\ &\quad +15\,(1-u)^3\,\mathbb{E}\left[T^3\right] - 2\,d\,u^2\mathbb{E}\left[T\,U^3\right] - 6\,d\,u\,(1-u)\,\mathbb{E}\left[T^2\,U\right] + d^2\,u\,\mathbb{E}\left[T^2\right]. \end{aligned}\tag{165}$$

Now, substitute the explicit moments obtained thus far. After factoring out $ab/15$, we get

$$\mathbb{E}_{\tau_{j-1,n}}\left[(A^3-d\alpha T)^2\right] = \frac{ab}{15}\,q_{a,b}(u), \tag{166}$$

where the function $q_{a,b}(\cdot)$ is defined as

$$\begin{aligned} q_{a,b}(u) \;\doteq\; & 15\,u^3\,P_3 + 75\,u^2\,(1-u)\,P_2 + 15\,u\,(1-u)^2\,P_1 + 15\,(1-u)^3\,P_0 + \\ & -10\,(a-b)^2\,(a+b)^2\,u^2 - 2\,(a-b)^2\,(7\,a^2+20\,a\,b+7\,b^2)\,u\,(1-u) \\ & +5\,(a-b)^2\,(a^2+3\,a\,b+b^2)\,u \end{aligned}\tag{167}$$

and P_3, P_2, P_1 and P_0 are the following polynomials in a and b:

$$\begin{aligned} P_3 &\doteq a^4 - a^3\,b + a^2\,b^2 - a\,b^3 + b^4, \\ P_2 &\doteq a^4 + a^3\,b - a^2\,b^2 + a\,b^3 + b^4, \\ P_1 &\doteq 7\,a^4 + 21\,a^3\,b + 19\,a^2\,b^2 + 21\,a\,b^3 + 7\,b^4, \\ P_0 &\doteq 3\,a^4 + 15\,a^3\,b + 25\,a^2\,b^2 + 15\,a\,b^3 + 3\,b^4. \end{aligned}$$

Expanding $q_{a,b}(u)$, and collecting the powers of u, one finds that the cubic terms cancel exactly and

$$\begin{aligned} q_{a,b}(u) = & \left(45a^4 + 225a^3b + 375a^2b^2 + 225ab^3 + 45b^4\right) \\ & - \left(39a^4 + 367a^3b + 808a^2b^2 + 367ab^3 + 39b^4\right)u \\ & + \left(4a^4 + 132a^3b + 448a^2b^2 + 132ab^3 + 4b^4\right)u^2. \end{aligned}$$

Summarizing, we can represent the function $q_{a,b}(\cdot)$ in the following form

$$q_{a,b}(u) = e_2\,(a,b)\,u^2 - e_1\,(a,b)\,u + e_0\,(a,b)\,,$$

with the coefficients $e_0(a,b)$, $e_1(a,b)$ and $e_2(a,b)$ given by

$$\begin{aligned} e_0(a,b) &\doteq 45\,a^4 + 225\,a^3\,b + 375\,a^2\,b^2 + 225\,a\,b^3 + 45\,b^4, \\ e_1(a,b) &\doteq 39\,a^4 + 367\,a^3\,b + 808\,a^2\,b^2 + 367\,a\,b^3 + 39\,b^4, \\ e_2(a,b) &\doteq 4\,a^4 + 132\,a^3\,b + 448\,a^2\,b^2 + 132\,a\,b^3 + 4\,b^4. \end{aligned}$$

Substituting Eq. (166) into Eq. (149) and using the original notation, we get

$$\mathbb{E}_{\tau_{j-1,n}}\left[\eta_{j,n}^2\right] = \frac{a\,b}{15\,n}\frac{\sigma^6_{\tau_{j-1,n}}}{\Gamma^3_{\tau_{j-1,n}}} q_{a,b}\left(\frac{\sigma'^2_{\tau_{j-1,n}}}{\Gamma_{\tau_{j-1,n}}}\right) + o_p(n^{-1}). \tag{168}$$

The final result is obtained by using Lemma 13, which gives

$$\sum_{\tau_{j,n}\le t} \mathbb{E}_{\tau_{j-1,n}}\left[\eta_{j,n}^2\right] \xrightarrow{p} \int_0^t \frac{\sigma_s^6}{15\,\Gamma_s^2} q_{a,b}\left(\frac{\sigma_s'^2}{\Gamma_s}\right) ds. \tag{169}$$

Next, we compute the limit of

$$C^{(W)}_{n,t} \doteq \sum_{\tau_{j,n}\le t} \mathbb{E}_{\tau_{j-1,n}}\left[\left(\xi_{j,n} + \xi'_{j,n}\right)\Delta_j^n W\right], \quad C^{(V)}_{n,t} \doteq \sum_{\tau_{j,n}\le t} \mathbb{E}_{\tau_{j-1,n}}\left[\left(\xi_{j,n} + \xi'_{j,n}\right)\Delta_j^n V\right].$$

Since $\xi'_{j,n} = O_p(n^{-1})$, we can simplify the above expressions into the following equivalent ones

$$C^{(W)}_{n,t} = \sum_{\tau_{j,n}\le t} \mathbb{E}_{\tau_{j-1,n}}\left[\xi_{j,n}\,\Delta_j^n W\right] + o_p(1), \quad C^{(V)}_{n,t} = \sum_{\tau_{j,n}\le t} \mathbb{E}_{\tau_{j-1,n}}\left[\xi_{j,n}\,\Delta_j^n V\right] + o_p(1).$$

We can simplify further by using the decomposition in Eq. (147) (which states that $\xi_{j,n} = \eta_{j,n} + r_{j,n}$ with $r_{j,n} = O_p(n^{-1})$). Hence, we may simply study the limit of the quantities

$$C^{(W)}_{n,t} = \sum_{\tau_{j,n}\le t} \mathbb{E}_{\tau_{j-1,n}}\left[\eta_{j,n}\,\Delta_j^n W\right] + o_p(1), \quad C^{(V)}_{n,t} = \sum_{\tau_{j,n}\le t} \mathbb{E}_{\tau_{j-1,n}}\left[\eta_{j,n}\,\Delta_j^n V\right] + o_p(1). \tag{170}$$

In line with the simplified notation used so far, it is convenient to set

$$A \doteq \alpha\,U - \beta\,C, \qquad B \doteq \beta\,U + \alpha\,C.$$

Then, the Brownian increments $\Delta_j^n W$ and $\Delta_j^n V$ are represented, respectively, as

$$\Delta_j^n W = \frac{A}{\sqrt{n}\sqrt{\Gamma_0}} + o_p(n^{-1/2}), \qquad \Delta_j^n V = \frac{B}{\sqrt{n}\sqrt{\Gamma_0}} + o_p(n^{-1/2}),$$

and, accordingly, the triangular array $\{\eta_{j,n}\}$ is

$$\eta_{j,n} = \frac{\sigma_0^3}{\sqrt{n}\,\Gamma_0^{3/2}}\left(A^3 - d\,\alpha\,T\right) + o_p(n^{-1/2}).$$

As a consequence, the product $\eta_{j,n}\,\Delta_j^n W$ has conditional expected value given by

$$\begin{aligned} \mathbb{E}_{\tau_{j-1,n}}\left[\eta_{j,n}\,\Delta_j^n W\right] &= \frac{\sigma_0^3}{n\,\Gamma_0^2}\,\mathbb{E}_{\tau_{j-1,n}}\left[\left(A^3 - d\,\alpha\,T\right)\,A\right] + o_p(n^{-1}), \\ \mathbb{E}_{\tau_{j-1,n}}\left[\eta_{j,n}\Delta_j^n V\right] &= \frac{\sigma_0^3}{n\,\Gamma_0^2}\,\mathbb{E}_{\tau_{j-1,n}}\left[\left(A^3 - d\,\alpha\,T\right)\,B\right] + o_p(n^{-1}). \end{aligned}$$

To compute $\mathbb{E}_{\tau_{j-1,n}}[(A^3 - d\alpha T)A]$ and $\mathbb{E}_{\tau_{j-1,n}}[(A^3 - d\,\alpha\,T)\,B]$ we need several new moments. Starting from the moments in Eq. (153) and the affine representation in Eq. (157), we obtain

$$\mathbb{E}\left[T\,U^2\right] = ab\,\mathbb{E}\left[T\right] + d\,\mathbb{E}\left[T\,U\right] = a^2\,b^2 + \frac{a\,b\,(a-b)^2}{3} = \frac{a\,b\,(a^2 + a\,b + b^2)}{3}.$$

Since we need $\mathbb{E}_{\tau_{j-1,n}}[A^4]$ and $\mathbb{E}_{\tau_{j-1,n}}[T\,A]$, we first notice that

$$A^4 = \alpha^4\,U^4 - 4\,\alpha^3\,\beta\,U^3\,C + 6\,\alpha^2\,\beta^2\,U^2\,C^2 - 4\,\alpha\,\beta^3\,U\,C^3 + \beta^4\,C^4$$
$$\Rightarrow \quad \mathbb{E}\left[A^4 \,\middle|\, \mathcal{F}_{t_{j-1}} \vee \sigma(T,U)\right] = \alpha^4\,U^4 + 6\,\alpha^2\,\beta^2\,U^2\,T + 3\beta^4\,T^2.$$

Therefore,

$$\mathbb{E}_{\tau_{j-1,n}}\left[A^4\right] = \alpha^4\,\mathbb{E}\left[U^4\right] + 6\,\alpha^2\,\beta^2\,\mathbb{E}\left[TU^2\right] + 3\,\beta^4\,\mathbb{E}\left[T^2\right].$$

Following the same logic, we have

$$\mathbb{E}\left[T\,A \,\middle|\, \mathcal{F}_{t_{j-1}} \vee \sigma(T,U)\right] = \alpha\,T\,U \Rightarrow \mathbb{E}_{\tau_{j-1,n}}[T\,A] = \alpha\,\mathbb{E}[T\,U].$$

Putting the last computations together, we obtain

$$\mathbb{E}_{\tau_{j-1,n}}\left[(A^3 - d\,\alpha\,T)\,A\right] = \mathbb{E}_{\tau_{j-1,n}}\left[A^4\right] - d\,\alpha\,\mathbb{E}_{\tau_{j-1,n}}[T\,A] = \alpha^4\,\mathbb{E}\left[U^4\right] + 6\,\alpha^2\,\beta^2\,\mathbb{E}\left[T\,U^2\right] + 3\,\beta^4\,\mathbb{E}\left[T^2\right] - d\,\alpha^2\,\mathbb{E}[T\,U].$$

Substituting the moments computed above yields

$$\mathbb{E}_{\tau_{j-1,n}}\left[(A^3 - d\,\alpha\,T)\,A\right] = ab\left(\alpha^4\,(a^2 - a\,b + b^2) + 2\,\alpha^2\,\beta^2\,(a^2 + a\,b + b^2) + \beta^4\,(a^2 + 3\,a\,b + b^2) - \frac{d^2}{3}\alpha^2\right).$$

Setting, as before, $u = \alpha^2$, and after some algebra, we obtain

$$\mathbb{E}_{\tau_{j-1,n}}\left[\left(A^3 - d\,\alpha\,T\right)\,A\right] = ab\left(a^2 + 3\,a\,b + b^2 - \frac{a^2 + 10\,a\,b + b^2}{3}\,u\right),$$

which gives (in the original notation)

$$\mathbb{E}_{\tau_{j-1,n}}\left[\eta_{j,n}\,\Delta_j^n W\right] = \frac{ab}{n}\,\frac{\sigma^3_{\tau_{j-1,n}}}{3\,\Gamma^3_{\tau_{j-1,n}}}\left(3\,\left(a^2 + 3\,a\,b + b^2\right)\,\Gamma_{\tau_{j-1,n}} - \left(a^2 + 10\,a\,b + b^2\right)\,\sigma'^2_{\tau_{j-1,n}}\right) + o_p(n^{-1}).$$

An application of Lemma 13 yields the required limit:

$$\sum_{\tau_{j,n}\le t} \mathbb{E}_{\tau_{j-1,n}}\left[\eta_{j,n}\Delta_j^n W\right] \xrightarrow{p} \int_0^t H_s\,ds,$$

where $\{H_s\}$ is the process defined as

$$H_s \doteq \frac{\sigma_s^3}{3\Gamma_s^2}\left(3\,(a^2 + 3\,a\,b + b^2)\,\Gamma_s - (a^2 + 10\,a\,b + b^2)\,\sigma'^2_s\right).$$

Similar computations show that

$$\sum_{\tau_{j,n}\le t} \mathbb{E}_{\tau_{j-1,n}}\left[\eta_{j,n}\Delta_j^n V\right] \xrightarrow{p} \int_0^t K_s\,ds,$$

with

$$K_s \doteq -\frac{\left(a^2 + 4ab + b^2\right)\,\sigma_s^3\,\sigma'_s\,\nu_s}{3\Gamma_s^2}.$$

□

Proof of Lemma 5. For every $j \ge 1$, let $S_j \doteq \text{sign}(P_j - P_{j-1})$ be the sign of the price change between $\tau_{j-1,n}$ and $\tau_{j,n}$. By construction (see Eq. 28) we have[25]

$$\begin{aligned} Y_j &= P_{j-1} + S_j\,h_{j-1}, \\ P_j &= P_{j-1} + S_j\,\alpha_n\,L_{j-1}, \end{aligned} \tag{171}$$

[25]Suppose, for example, that $\omega \in \Omega$ is such that the process Y hits the upper barrier, that is $Y_j = P_{j-1} + \alpha_n\,(L_{j-1} - 1/2 + \eta)$ then we have $P_j = \alpha_n\ \text{Round}\left(\frac{Y_j}{\alpha_n}\right) = \alpha_n\ \text{Round}\left(\frac{P_{j-1}}{\alpha_n} + L_{j-1} - \frac{1}{2} + \eta\right) = \alpha_n\left(\frac{P_{j-1}}{\alpha_n} + L_{j-1}\right) = P_{j-1} + \alpha_n\,L_{j-1} > P_{j-1}$, where, in the second-to-last identity, we have used the fact that $P_{j-1}/\alpha_n \in \mathbb{N}$, $L_{j-1} \in \mathbb{N}$ and $\eta - 1/2 \in (-1/2, 1/2)$. In particular we have $S_j = \text{sign}(P_j - P_{j-1}) = 1$ and, accordingly, the two Eqs. (171) are verified. The same reasoning applies if $\omega \in \Omega$ is such that the process Y hits the lower barrier.

and so

$$z_j = Y_j - P_j = S_j\,\left(h_{j-1} - \alpha_n\,L_{j-1}\right) = -S_j\,\alpha_n\,\left(\frac{1}{2} - \eta\right), \tag{172}$$

which gives the identity

$$h_j^2 - z_j^2 = \alpha_n^2\left(\left(L_j - \left(\frac{1}{2} - \eta\right)\right)^2 - \left(\frac{1}{2} - \eta\right)^2\right) = \alpha_n^2\,L_j\,\left(L_j - 1 + 2\,\eta\right). \tag{173}$$

Fix, now, $j \geq 2$ and apply this identity at index $j-1$. Trivially, we have $Q_{j-1}(\tau_{j-1,n}) = z_{j-1}$, $Q_{j-1}(\tau_{j,n}) \in \{-h_{j-1}, h_{j-1}\}$, $G_{j-1}(\tau_{j-1,n}) = h_{j-1}^2 - z_{j-1}^2$, and $G_{j-1}(\tau_{j,n}) = 0$. Consider the function

$$g\,(q,\sigma) \doteq \frac{h_{j-1}^2 - q^2}{\sigma^2},$$

which has the properties

$$g\,(\pm h_{j-1},\sigma) = 0, \quad \frac{1}{2}\sigma^2\,\frac{\partial^2 g}{\partial q^2} = -1.$$

Its partial derivatives are

$$g_q = -\frac{2q}{\sigma^2}, \quad g_{qq} = -\frac{2}{\sigma^2}, \quad g_\sigma = -\frac{2(h_{j-1}^2 - q^2)}{\sigma^3}, \quad g_{\sigma\sigma} = \frac{6(h_{j-1}^2 - q^2)}{\sigma^4}, \quad g_{q\sigma} = \frac{4q}{\sigma^3}.$$

For $t \in [\tau_{j-1,n}, \tau_{j,n}]$, we have $dQ_{j-1}(t) = \mu_t^{(Y)}\,dt + \sigma_t^{(Y)}\,dW_t$, so

$$d[Q_{j-1}]_t = \left(\sigma_t^{(Y)}\right)^2\,dt, \quad d[\sigma^{(Y)}]_t = ((\sigma_t'^{(Y)})^2 + (\nu_t^{(Y)})^2)dt, \quad d[Q_{j-1},\sigma^{(Y)}]_t = \sigma_t^{(Y)}\sigma_t'^{(Y)}dt.$$

Itô's formula, therefore, yields

$$\begin{aligned} dg(Q_{j-1}(t),\sigma_t^{(Y)}) &= g_q\,dQ_{j-1}(t) + g_\sigma\,d\sigma_t^{(Y)} + \frac{1}{2}g_{qq}(\sigma_t^{(Y)})^2dt + \frac{1}{2}g_{\sigma\sigma}((\sigma_t'^{(Y)})^2 + (\nu_t^{(Y)})^2)dt + g_{q\sigma}\sigma_t^{(Y)}\sigma_t'^{(Y)}dt \\ &= -dt + \Phi_{j-1}^{(Y;W)}(t)\,dW_t + \Phi_{j-1}^{(Y;V)}(t)\,dV_t + \Phi_{j-1}^{(Y;\Delta)}(t)\,dt, \end{aligned}$$

where $\Phi_{j-1}^{(Y;W)}(t)$, $\Phi_{j-1}^{(Y;V)}(t)$ and $\Phi_{j-1}^{(Y;\Delta)}(t)$ are given in Eq. (31). Integrating from $\tau_{j-1,n}$ to $\tau_{j,n}$, using the identity in Eq. (173) and the following

$$g(Q_{j-1}(\tau_{j,n}),\sigma_{\tau_{j,n}}^{(Y)}) = 0, \qquad g(Q_{j-1}(\tau_{j-1,n}),\sigma_{\tau_{j-1,n}}^{(Y)}) = \frac{h_{j-1}^2 - z_{j-1}^2}{\left(\sigma_{\tau_{j-1,n}}^{(Y)}\right)^2},$$

yields the representation in Eq. (30). Finally, the probability orders in Eq. (32) follow from the geometry of the barriers. Since $L_{j-1} \in \{1,\ldots,m\}$, uniformly in $t \in [\tau_{j-1,n}, \tau_{j,n}]$,

$$\left|Q_{j-1}\,(t)\right| \leq h_{j-1} \leq \alpha_n\left(m - \frac{1}{2} + \eta\right), \qquad 0 \leq G_{j-1}\,(t) \leq h_{j-1}^2 \leq \alpha_n^2\left(m - \frac{1}{2} + \eta\right)^2.$$

□

Proof of Theorem 4.1. From Eq. (171) we derive

$$\left(\log\left(P_j\right) - \log\left(P_{j-1}\right)\right)^2 = \left(\log\left(1 + \frac{S_j\alpha_n L_{j-1}}{P_{j-1}}\right)\right)^2.$$

Taylor's formula therefore gives, uniformly over j with $\tau_{j,n} \leq t$,

$$\log\left(P_j\right) - \log\left(P_{j-1}\right) = \frac{S_j\alpha_n L_{j-1}}{P_{j-1}} - \frac{\alpha_n^2\,L_{j-1}^2}{2P_{j-1}^2} + O_p\left(\alpha_n^3\right).$$

Since $S_j^2 = 1$, we get

$$\left(\log\left(P_j\right) - \log\left(P_{j-1}\right)\right)^2 = \frac{\alpha_n^2 L_{j-1}^2}{P_{j-1}^2} - S_j \frac{\alpha_n^3 L_{j-1}^3}{P_{j-1}^3} + O_p\left(\alpha_n^4\right) = \frac{\alpha_n^2 L_{j-1}^2}{P_{j-1}^2} + O_p\left(\alpha_n^3\right). \tag{174}$$

Expressing L_{j-1}^2 as

$$L_{j-1}^2 = L_{j-1}\left(L_{j-1} - 1 + 2\eta + 1 - 2\eta\right) = L_{j-1}\left(L_{j-1} - 1 + 2\eta\right) + (1 - 2\eta) L_{j-1},$$

we obtain

$$\left(\log\left(P_j\right) - \log\left(P_{j-1}\right)\right)^2 = \frac{\alpha_n^2 L_{j-1}\left(L_{j-1} - 1 + 2\eta\right)}{P_{j-1}^2} + (1 - 2\eta)\frac{\alpha_n^2 L_{j-1}}{P_{j-1}^2} + O_p\left(\alpha_n^3\right). \tag{175}$$

Since, pathwise, $|Y_{j-1} - P_{j-1}| \leq \alpha_n/2$, we have $P_{j-1} = Y_{j-1} + O_p(\alpha_n)$ hence

$$\frac{\alpha_n^2 L_{j-1}\left(L_{j-1} - 1 + 2\eta\right)}{P_{j-1}^2} = \frac{\alpha_n^2 L_{j-1}\left(L_{j-1} - 1 + 2\eta\right)}{Y_{j-1}^2} + O_p\left(\alpha_n^3\right).$$

Combining this last approximation with the chaotic representation of durations in Eq. (30), we have

$$\begin{aligned}
\frac{\alpha_n^2 L_{j-1}\left(L_{j-1} - 1 + 2\eta\right)}{P_{j-1}^2} &= \sigma_{\tau_{j-1,n}}^2 \Delta_{j,n} - \mathcal{R}_{j,n} + O_p\left(\alpha_n^3\right), \\
\mathcal{R}_{j,n} &\doteq \sigma_{\tau_{j-1,n}}^2 \left(\int_{\tau_{j-1,n}}^{\tau_{j,n}} \Phi_{j-1}^{(Y;W)}(t)\, dW_t \right. \\
&\quad \left. + \int_{\tau_{j-1,n}}^{\tau_{j,n}} \Phi_{j-1}^{(Y;V)}(t)\, dV_t + \int_{\tau_{j-1,n}}^{\tau_{j,n}} \Phi_{j-1}^{(Y;\Delta)}(t)\, dt \right).
\end{aligned} \tag{176}$$

From the order of probability in Eq. (32), we also have

$$\sum_{\substack{j\geq 1 \\ t_{j,n}\leq t}} \mathbb{E}_{\tau_{j-1,n}}\left[\mathcal{R}_{j,n}\right] = \sum_{\substack{j\geq 1 \\ t_{j,n}\leq t}} \sigma_{\tau_{j-1,n}}^2 \mathbb{E}_{\tau_{j-1,n}}\left[\int_{\tau_{j-1,n}}^{\tau_{j,n}} \Phi_{j-1}^{(Y;\Delta)}(t)\, dt\right] = O_p\left(\alpha_n\right) \xrightarrow{p} 0,$$

where we have used $\Phi_{j-1}^{(Y;\Delta)}(t) = O_p(\alpha_n)$, $\Delta_{j,n} = O_p(\alpha_n^2)$ and $N_{t,n} = O_p(\alpha_n^{-2})$. Eq. (175) thus gives

$$\sum_{\substack{j\geq 1 \\ t_{j,n}\leq t}} \left(\log P_j - \log P_{j-1}\right)^2 = \sum_{\substack{j\geq 1 \\ t_{j,n}\leq t}} \sigma_{\tau_{j-1,n}}^2 \Delta_{j,n} + (1 - 2\eta)\, \alpha_n^2 \sum_{\substack{j\geq 1 \\ t_{j,n}\leq t}} \frac{L_{j-1}}{P_{j-1}^2} + O_p\left(\alpha_n\right),$$

which proves Eq. (33).

To prove Eq. (34), first define

$$\mathcal{S}_n(t) \doteq (1 - 2\eta)\, \alpha_n^2 \sum_{\substack{j\geq 1 \\ t_{j,n}\leq t}} \frac{L_{j-1}}{P_{j-1}^2}.$$

We have the following identity

$$\begin{aligned}
\mathcal{S}_n(t) &= \alpha_n^2 \sum_{\substack{j\geq 1 \\ t_{j,n}\leq t}} \frac{L_{j-1}^2 - L_{j-1}\left(L_{j-1} - 1 + 2\eta\right)}{L_{j-1}\left(L_{j-1} - 1 + 2\eta\right)} \frac{L_{j-1}\left(L_{j-1} - 1 + 2\eta\right)}{P_{j-1}^2} \\
&= \sum_{\substack{j\geq 1 \\ t_{j,n}\leq t}} \frac{L_{j-1}^2 - L_{j-1}\left(L_{j-1} - 1 + 2\eta\right)}{L_{j-1}\left(L_{j-1} - 1 + 2\eta\right)} \left(\sigma_{\tau_{j-1,n}}^2 \Delta_{j,n} - \mathcal{R}_{j,n}\right) + o_p(1), \\
&= \sum_{\substack{j\geq 1 \\ t_{j,n}\leq t}} \frac{L_{j-1}^2 - L_{j-1}\left(L_{j-1} - 1 + 2\eta\right)}{L_{j-1}\left(L_{j-1} - 1 + 2\eta\right)} \sigma_{\tau_{j-1,n}}^2 \Delta_{j,n} - (1 - 2\eta) \sum_{\substack{j\geq 1 \\ t_{j,n}\leq t}} \frac{L_{j-1}}{L_{j-1}\left(L_{j-1} - 1 + 2\eta\right)} \mathcal{R}_{j,n} + o_p(1),
\end{aligned}$$

where we have used Eq. (176). We obtain

$$\sum_{\substack{j\geq 1\\ t_{j,n}\leq t}} \frac{L_{j-1}}{L_{j-1}\left(L_{j-1}-1+2\eta\right)}\mathcal{R}_{j,n} \xrightarrow{p} 0,$$

as the coefficients that multiply $\mathcal{R}_{j,n}$ are bounded and $\mathcal{F}_{\tau_{j-1,n}}$-measurable. Define $a(k) \doteq k\left(k-1+2\eta\right)$. Now consider the difference

$$\mathcal{D}_n(t) \doteq \sum_{\substack{j\geq 1\\ t_{j,n}\leq t}} \left(\frac{L_{j-1}^2 - L_{j-1}\left(L_{j-1}-1+2\eta\right)}{L_{j-1}\left(L_{j-1}-1+2\eta\right)} - q\left(\chi_{\tau_{j-1,n}}\right)\right)\sigma^2_{\tau_{j-1,n}}\Delta_{j,n} = \sum_{\substack{j\geq 1\\ t_{j,n}\leq t}} Z_{j,n}$$

where $q(x) \doteq (m_2(x) - m_\Delta(x))/m_\Delta(x)$ and the last identity defines the triangular array $\{Z_{j,n}\}$. Since both L_{j-1} and $\chi_{\tau_{j-1,n}}$ are $\mathcal{F}_{\tau_{j-1,n}}$-measurable, we have

$$\begin{aligned}
\mathbb{E}_{\tau_{j-1,n}}\left[Z_{j,n}\right] &= \left(\frac{L_{j-1}^2 - a\left(L_{j-1}\right)}{a\left(L_{j-1}\right)} - q\left(\chi_{\tau_{j-1,n}}\right)\right)\mathbb{E}_{\tau_{j-1,n}}\left[\sigma^2_{\tau_{j-1,n}}\Delta_{j,n}\right] \\
&= \left(\frac{L_{j-1}^2 - a\left(L_{j-1}\right)}{a\left(L_{j-1}\right)} - q\left(\chi_{\tau_{j-1,n}}\right)\right)\frac{\alpha_n^2\, a\left(L_{j-1}\right)}{P_{j-1}^2} + O_p\left(\alpha_n^3\right) \\
&= Z'_{j-1,n} + O_p\left(\alpha_n^3\right), \qquad (177)
\end{aligned}$$

where the last identity defines the triangular array $\{Z'_{j-1,n}\}$. By construction, $Z'_{j-1,n}$ is $\mathcal{H}_{j,n}$-measurable and

$$\mathbb{E}\left[Z'_{j-1,n}\,\middle|\,\mathcal{H}_{j-1,n}\right] = 0.$$

We thus have the decomposition

$$Z_{j,n} = \left(Z_{j,n} - \mathbb{E}_{\tau_{j-1,n}}\left[Z_{j,n}\right]\right) + \left(\mathbb{E}_{\tau_{j-1,n}}\left[Z_{j,n}\right] - Z'_{j-1,n}\right) + Z'_{j-1,n} = \left(Z_{j,n} - \mathbb{E}_{\tau_{j-1,n}}\left[Z_{j,n}\right]\right) + Z'_{j-1,n} + O_p\left(\alpha_n^3\right).$$

It is now easy to see that[26]

$$\begin{aligned}
\sum_{\substack{j\geq 1\\ t_{j,n}\leq t}} \mathbb{E}_{\tau_{j-1,n}}\left[\left(Z_{j,n} - \mathbb{E}_{\tau_{j-1,n}}\left[Z_{j,n}\right]\right)^2\right] &\xrightarrow{p} 0, \\
\sum_{\substack{j\geq 1\\ t_{j,n}\leq t}} \mathbb{E}\left[\left(Z'_{j-1,n}\right)^2\,\middle|\,\mathcal{H}_{j-1,n}\right] &\xrightarrow{p} 0.
\end{aligned}$$

which implies that $\mathcal{D}_n(t) \xrightarrow[n\to\infty]{\text{ucp}} 0$ and, accordingly, proves Eq. (34). □

Proof of Corollary 4.2. The result is an immediate consequence of Eq. (36) and the definitions in Eq. (35). In particular, if $L_j = 1$ for every $j \geq 0$, then $m_2(x) = 1$ and $m_\Delta(x) = 2\eta$. □

[26] Since the coefficient multiplying $\Delta_{j,n}$ in $Z_{j,n}$ is bounded and $\mathbb{E}_{\tau_{j-1,n}}[\Delta^2_{j,n}] \leq C\alpha_n^4$, we have

$$\mathbb{E}_{\tau_{j-1,n}}\left[\left(Z_{j,n} - \mathbb{E}_{\tau_{j-1,n}}\left[Z_{j,n}\right]\right)^2\right] \leq \mathbb{E}_{\tau_{j-1,n}}\left[Z^2_{j,n}\right] \leq C\alpha_n^4.$$

Therefore, writing $N_n(t) \doteq \#\{j \geq 1 \mid t_{j,n} \leq t\} = O_p(\alpha_n^{-2})$, we obtain

$$0 \leq \sum_{\substack{j\geq 1\\ t_{j,n}\leq t}} \mathbb{E}_{\tau_{j-1,n}}\left[\left(Z_{j,n} - \mathbb{E}_{\tau_{j-1,n}}\left[Z_{j,n}\right]\right)^2\right] \leq C\alpha_n^4 N_n(t) = O_p\left(\alpha_n^2\right) \xrightarrow{p} 0.$$

Moreover, the definition of $Z'_{j-1,n}$ gives $|Z'_{j-1,n}| \leq C\alpha_n^2$, and, consequently,

$$0 \leq \sum_{\substack{j\geq 1\\ t_{j,n}\leq t}} \mathbb{E}\left[\left(Z'_{j-1,n}\right)^2\,\middle|\,\mathcal{H}_{j-1,n}\right] \leq C\alpha_n^4 N_n(t) = O_p\left(\alpha_n^2\right) \xrightarrow{p} 0.$$

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